816.

ON THE BITANGENTS OF A PLANE QUARTIC.

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RIEMANN in the paper "Zur Theorie der *Abel*schen Functionen für den Fall $p=3$," *Werke*, pp. 456—479, has given a remarkably elegant solution of the problem of the bitangents of a plane quartic. But his formulæ may be improved by a slight change; viz. we may in his first equation $x+y+z+\xi+\eta+\zeta=0$ introduce coefficients so as to bring this into the same form as the other three equations. It thus appears that, instead of his $3+3$ equations of the forms

$$\frac{x}{1-\beta\gamma}+\frac{y}{1-\gamma\alpha}+\frac{z}{1-\alpha\beta}=0 \text{ and } \frac{\xi}{\alpha(\gamma-\beta)}+\frac{\eta}{\beta(\gamma-\alpha)}+\frac{\zeta}{\gamma(\beta-\alpha)}=0$$

respectively, we have $6+6$ equations of like forms; but these two systems each of 6 equations are equivalent to each other, so that instead of $6+16+3+3=28$, we have $6+16+6(=6)=28$ equations for the 28 bitangents*. I make another slight change of notation by introducing the single letters f, g, h to denote the reciprocals $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$; and I consider the whole question as follows. The theory is based on the equations

$$\begin{aligned}
a\,x+b\,y+c\,z+f\,\xi+g\,\eta+h\,\zeta&=0,\\
a_1x+b_1y+c_1z+f_1\xi+g_1\eta+h_1\zeta&=0,\\
a_2x+b_2y+c_2z+f_2\xi+g_2\eta+h_2\zeta&=0,\\
a_3x+b_3y+c_3z+f_3\xi+g_3\eta+h_3\zeta&=0,
\end{aligned}$$

where $af=bg=ch=a_1f_1=\&c.=c_3h_3=1$: and the coefficients $a, b, c, a_1, b_1, c_1, a_2, b_2, c_2$ are arbitrary. The equations $x=0,\ y=0,\ z=0$ represent any three given lines: and

* It will appear further on that the equation of each of the last-mentioned 6 bitangents can be expressed in 8 different forms.

considering ξ, η, ζ to be determined as linear functions of x, y, z by means of the first three equations, then (the coefficients a, b, c, a_1, b_1, c_1, a_2, b_2, c_2 being determined accordingly) the equations $\xi=0$, $\eta=0$, $\zeta=0$ will also represent any three given lines. But observe that, if the equation of the first of these lines is $x+my+nz=0$, ξ is not = an arbitrary multiple $\theta(x+my+nz)$ of the linear function $x+my+nz$, but the constant factor θ has a completely determinate value: and the like as regards η and ζ respectively.

The coefficients a_3, b_3, c_3 of the fourth equation are such that this fourth equation is a mere consequence of the other three: viz. we must have

$$\begin{aligned}
a_3 &= \lambda a + \lambda_1 a_1 + \lambda_2 a_2,\\
b_3 &= \lambda b + \lambda_1 b_1 + \lambda_2 b_2,\\
c_3 &= \lambda c + \lambda_1 c_1 + \lambda_2 c_2,\\
f_3 &= \lambda f + \lambda_1 f_1 + \lambda_2 f_2,\\
g_3 &= \lambda g + \lambda_1 g_1 + \lambda_2 g_2,\\
h_3 &= \lambda h + \lambda_1 h_1 + \lambda_2 h_2,
\end{aligned}$$

or, what is the same thing, we must have

$$\left\|\begin{matrix} a, & b, & c, & f, & g, & h\\ a_1, & b_1, & c_1, & f_1, & g_1, & h_1\\ a_2, & b_2, & c_2, & f_2, & g_2, & h_2\\ a_3, & b_3, & c_3, & f_3, & g_3, & h_3\end{matrix}\right\| = 0,$$

viz. each of the determinants formed with four columns out of this matrix is equal to 0.

Using the equations in λ, λ_1, λ_2, and writing for shortness

$$\begin{aligned}
&a_1f_2+a_2f_1,\quad a_2f+af_2,\quad af_1+a_1f = F,\ F_1,\ F_2,\\
&b_1g_2+b_2g_1,\quad b_2g+bg_2,\quad bg_1+b_1g = G,\ G_1,\ G_2,\\
&c_1h_2+c_2h_1,\quad c_2h+ch_2,\quad ch_1+c_1h = H,\ H_1,\ H_2;
\end{aligned}$$

then, forming the products a_3f_3, b_3g_3, c_3h_3, we find

$$\begin{aligned}
1-\lambda^2-\lambda_1^2-\lambda_2^2 &= F\lambda_1\lambda_2 + F_1\lambda_2\lambda + F_2\lambda\lambda_1,\\
\text{,,}\qquad &= G\lambda_1\lambda_2 + G_1\lambda_2\lambda + G_2\lambda\lambda_1,\\
\text{,,}\qquad &= H\lambda_1\lambda_2 + H_1\lambda_2\lambda + H_2\lambda\lambda_1,
\end{aligned}$$

or, as these equations may be written,

$$\begin{gathered}
1-\lambda^2-\lambda_1^2-\lambda_2^2 \ :\ \lambda_1\lambda_2 \ :\ \lambda_2\lambda \ :\ \lambda\lambda_1\\
= \begin{vmatrix} F, & F_1, & F_2\\ G, & G_1, & G_2\\ H, & H_1, & H_2\end{vmatrix} : \begin{vmatrix} 1, & F_1, & F_2\\ 1, & G_1, & G_2\\ 1, & H_1, & H_2\end{vmatrix} : \begin{vmatrix} 1, & F_2, & F\\ 1, & G_2, & G\\ 1, & H_2, & H\end{vmatrix} : \begin{vmatrix} 1, & F, & F_1\\ 1, & G, & G_1\\ 1, & H, & H_1\end{vmatrix},
\end{gathered}$$

say for shortness

$$= \Pi : \Lambda : \Lambda_1 : \Lambda_2.$$

We have thus

$$\frac{\lambda^2}{1-\lambda^2-\lambda_1^2-\lambda_2^2} = \frac{\Lambda_1\Lambda_2}{\Pi\Lambda},$$

$$\frac{\lambda_1^2}{1-\lambda^2-\lambda_1^2-\lambda_2^2} = \frac{\Lambda_2\Lambda}{\Pi\Lambda_1},$$

$$\frac{\lambda_2^2}{1-\lambda^2-\lambda_1^2-\lambda_2^2} = \frac{\Lambda\Lambda_1}{\Pi\Lambda_2},$$

and thence

$$\frac{1}{1-\lambda^2-\lambda_1^2-\lambda_2^2} = \frac{1}{\Pi}\left(\frac{\Lambda_1\Lambda_2}{\Lambda} + \frac{\Lambda_2\Lambda}{\Lambda_1} + \frac{\Lambda\Lambda_1}{\Lambda_2}\right),$$

equations which give λ, λ_1, λ_2, and consequently a_3, b_3, c_3, f_3, g_3, h_3 in terms of known quantities, the several expressions depending on the single radical

$$\sqrt{\frac{1}{\Pi}\left(\frac{\Lambda_1\Lambda_2}{\Lambda} + \frac{\Lambda_2\Lambda}{\Lambda_1} + \frac{\Lambda\Lambda_1}{\Lambda_2}\right)}.$$

Observe that we have rationally

$$\lambda : \lambda_1 : \lambda_2 = \frac{1}{\Lambda} : \frac{1}{\Lambda_1} : \frac{1}{\Lambda_2}.$$

I consider now the quartic curve

$$\sqrt{x\xi} + \sqrt{y\eta} + \sqrt{z\zeta} = 0,$$

and I write down the equations of the 28 bitangents, each with its triple-theta characteristic as given by Riemann, and also with its duad symbol, derived from Hesse's method. I assume that these characteristics and duad symbols are properly attached to the several lines: and I insert also the current nos. 1 to 28. The equations are:

Current No.	Duad Symbol	Characteristic	
1	18	111 111	$x=0$,
2	28	001 011	$y=0$,
3	38	011 001	$z=0$,
4	23	010 010	$\xi=0$,
5	13	100 110	$\eta=0$,
6	12	110 100	$\zeta=0$,

Current No.	Duad Symbol	Characteristic	
7	48	101 100	$ax+by+cz=0,\quad f\xi+g\eta+h\zeta=0,$
8	14	010 011	$f\xi+by+cz=0,\quad ax+g\eta+h\zeta=0,$
9	24	100 111	$ax+g\eta+cz=0,\quad f\xi+by+h\zeta=0,$
10	34	110 101	$ax+by+h\zeta=0,\quad f\xi+g\eta+cz=0,$
11	58	100 101	$a_1x+b_1y+c_1z=0,$
12	15	011 010	$f_1\xi+b_1y+c_1z=0,$
13	25	101 110	$a_1x+g_1\eta+c_1z=0,$
14	35	111 100	$a_1x+b_1y+h_1\zeta=0,$
15	68	110 010	$a_2x+b_2y+c_2z=0,$
16	16	001 101	$f_2\xi+b_2y+c_2z=0,$
17	26	111 001	$a_2x+g_2\eta+c_2z=0,$
18	36	101 011	$a_2x+b_2y+h_2\zeta=0,$
19	78	010 110	$a_3x+b_3y+c_3z=0,$
20	17	101 001	$f_3\xi+b_3y+c_3z=0,$
21	27	011 101	$a_3x+g_3\eta+c_3z=0,$
22	37	000 111	$a_3x+b_3y+h_3\zeta=0,$
23	67	100 100	$\frac{x}{bc-b_1c_1}+\frac{y}{ca-c_1a_1}+\frac{z}{ab-a_1b_1}=0,\quad \frac{\xi}{g_2h_2-g_3h_3}+\frac{\eta}{h_2f_2-h_3f_3}+\frac{\zeta}{f_2g_2-f_3g_3}=0,$
			$\frac{x}{bc-b_1c_1}+\frac{\eta}{bb_1(ca_1-c_1a)}-\frac{\zeta}{cc_1(ab_1-a_1b)}=0,\quad \frac{\xi}{g_2h_2-g_3h_3}+\frac{y}{g_2g_3(h_2f_3-h_3f_2)}-\frac{z}{h_2h_3(f_2g_3-f_3g_2)}=0,$
			$\frac{-\xi}{aa_1(bc_1-b_1c)}+\frac{y}{ca-c_1a_1}+\frac{\zeta}{cc_1(ab_1-a_1b)}=0,\quad \frac{-x}{f_2f_3(g_2h_3-g_3h_2)}+\frac{\eta}{h_2f_2-h_3f_3}+\frac{z}{h_2h_3(f_2g_3-f_3g_2)}=0,$
			$\frac{\xi}{aa_1(bc_1-b_1c)}-\frac{\eta}{bb_1(ca_1-c_1a)}+\frac{z}{ab-a_1b_1}=0,\quad \frac{x}{f_2f_3(g_2h_3-g_3h_2)}-\frac{y}{g_2g_3(h_2f_3-h_3f_2)}+\frac{\zeta}{f_2g_2-f_3g_3}=0,$
24	57	110 011	$\frac{x}{bc-b_2c_2}+\frac{y}{ca-c_2a_2}+\frac{z}{ab-a_2b_2}=0,$
25	56	010 111	$\frac{x}{bc-b_3c_3}+\frac{y}{ca-c_3a_3}+\frac{z}{ab-a_3b_3}=0,$
26	45	001 001	$\frac{x}{b_2c_2-b_3c_3}+\frac{y}{c_2a_2-c_3a_3}+\frac{z}{a_2b_2-a_3b_3}=0,$
27	46	011 110	$\frac{x}{b_3c_3-b_1c_1}+\frac{y}{c_3a_3-c_1a_1}+\frac{z}{a_3b_3-a_1b_1}=0,$
28	47	111 010	$\frac{x}{b_1c_1-b_2c_2}+\frac{y}{c_1a_1-c_2a_2}+\frac{z}{a_1b_1-a_2b_2}=0.$

As regards each of the bitangents 7 to 22, the equivalence of the two forms of equation is at once seen by means of the fundamental equations $ax+by+cz+f\xi+g\eta+h\zeta=0$: as regards the remaining bitangents 23 to 28, the equation of each of these is expressible in eight different forms, which are written down in full for the bitangent 23; the equivalence of the eight forms of equation must of course be proved.

I proceed to show that each of the 28 lines is, in fact, a bitangent to the quartic curve

$$\sqrt{x\xi}+\sqrt{y\eta}+\sqrt{z\zeta}=0,$$

or, what is the same thing, that writing this equation in the form

$$\Omega = x^2\xi^2+y^2\eta^2+z^2\zeta^2-2yz\eta\zeta-2zx\zeta\xi-2xy\xi\eta=0,$$

then that by means of any one of these equations Ω becomes a perfect square.

This is obviously the case for each of the equations 1, 2, 3, 4, 5, 6.

For the equations 7, 8, 9, 10, observe that the equation of the quartic may be written:

$$\sqrt{axf\xi}+\sqrt{byg\eta}+\sqrt{czh\zeta}=0,$$

or, what is the same thing,

$$\Omega = a^2x^2 . f^2\xi^2+\ldots=0.$$

Hence writing x, y, z, ξ, η, ζ in place of ax, by, cz, $f\xi$, $g\eta$, $h\zeta$, so that now

$$x+y+z+\xi+\eta+\zeta=0,$$

the equation of the line 7 is expressed in the two equivalent forms $x+y+z=0$, $\xi+\eta+\zeta=0$. We have for Ω its original value

$$\begin{aligned}\Omega &= x^2\xi^2+y^2\eta^2+z^2\zeta^2-2yz\eta\zeta-2zx\zeta\xi-2xy\xi\eta,\\ &=(z\zeta-x\xi-y\eta)^2-4xy\xi\eta;\end{aligned}$$

and writing herein $z=-x-y$, $\zeta=-\xi-\eta$, we have $z\zeta-x\xi-y\eta=x\eta+y\xi$ and consequently $\Omega=(x\eta+y\xi)^2-4xy\xi\eta$, $=(x\eta-y\xi)^2$, a perfect square. The like proof applies to the equations 8, 9, 10: and then clearly the result applies to the equations 11, 12, 13, 14; 15, 16, 17, 18; 19, 20, 21, 22.

We come now to the equation of the line 23, say this is in the first instance taken to be

$$\frac{x}{bc-b_1c_1}+\frac{y}{ca-c_1a_1}+\frac{z}{ab-a_1b_1}=0;$$

if from this equation, by means of the two fundamental equations

$$\begin{aligned}ax+by+cz+f\xi+g\eta+h\zeta&=0,\\ a_1x+b_1y+c_1z+f_1\xi+g_1\eta+h_1\zeta&=0,\end{aligned}$$

we eliminate first (y, z), secondly (z, x), and thirdly (x, y), it is found that ξ, η, ζ will also disappear in the three cases respectively, and that the resulting equations are

$$\frac{x}{bc-b_1c_1} + \frac{\eta}{bb_1(ca_1-c_1a)} - \frac{\zeta}{cc_1(ab_1-a_1b)} = 0,$$

$$-\frac{\xi}{aa_1(bc_1-b_1c)} + \frac{y}{ca-c_1a_1} + \frac{\zeta}{cc_1(ab_1-a_1b)} = 0,$$

$$\frac{\xi}{aa_1(bc_1-b_1c)} - \frac{\eta}{bb_1(ca_1-c_1a)} + \frac{z}{ab-a_1b_1} = 0,$$

so that each of these is, in fact, equivalent to the original equation in (x, y, z): and observe that, adding together the three equations, we reproduce the original equation in (x, y, z).

Write for shortness

$$\begin{aligned} P &= bc-b_1c_1, & \alpha &= bc_1-b_1c, \\ Q &= ca-c_1a_1, & \beta &= ca_1-c_1a, \\ R &= ab-a_1b_1, & \gamma &= ab_1-a_1b, \end{aligned}$$

then the equation in (x, y, z) is

$$QRx+RPy+PQz=0,$$

so that, eliminating y and z, we find

$$\begin{vmatrix} b\ , & c\ , & ax+f\xi+g\eta+h\zeta \\ b_1\ , & c_1\ , & a_1x+f_1\xi+g_1\eta+h_1\zeta \\ RP, & PQ, & QRx \end{vmatrix} = 0.$$

In this equation, the coefficient of x is

$$\begin{aligned} & (bc_1-b_1c)\,QR+(ca_1-c_1a)\,RP+(ab_1-a_1b)\,PQ, \\ = \ & (bc_1-b_1c)\,\{a^2bc+a_1^2b_1c_1-aa_1(bc_1+b_1c)\} \\ & +(ca_1-c_1a)\,\{ab^2c+a_1b_1^2c_1-bb_1(ca_1+c_1a)\} \\ & +(ab_1-a_1b)\,\{abc^2+a_1b_1c_1^2-cc_1(ab_1+a_1b)\}, \\ = & -aa_1(b^2c_1^2-b_1^2c^2)-bb_1(c^2a_1^2-c_1^2a^2)-cc_1(a^2b_1^2-a_1^2b^2), \\ = & -(bc_1-b_1c)(ca_1-c_1a)(ab_1-a_1b), \\ = & -\alpha\beta\gamma. \end{aligned}$$

The coefficient of ξ is

$$= RP(cf_1-c_1f)-PQ(bf_1-b_1f),$$

which $\left(\text{observing that } cf_1-c_1f, =\frac{c}{a_1}-\frac{c_1}{a}, \text{ is } =\frac{1}{aa_1}Q, \text{ and } bf_1-b_1f, =\frac{b}{a_1}-\frac{b_1}{a}, \text{ is } =\frac{1}{aa_1}R\right)$ is $=0$.

The coefficient of η is

$$RP(cg_1-c_1g)-PQ(bg_1-b_1g),$$

which is

$$= \quad RP\frac{1}{bb_1}(bc-b_1c_1) - PQ\frac{1}{bb_1}(b^2-b_1^2),$$

$$= \frac{P}{bb_1}\{(bc-b_1c_1)(ab-a_1b_1)-(b^2-b_1^2)(ca-c_1a_1)\},$$

$$= \frac{P}{bb_1}\{b_1^2ca+b^2c_1a_1-bb_1(ac_1+a_1c)\}$$

$$= -\frac{P}{bb_1}\gamma\alpha\,;$$

and similarly the coefficient of ζ is

$$= \quad \frac{P}{cc_1}\alpha\beta.$$

We have thus in x, η, ζ the equation

$$-\alpha\beta\gamma x - \frac{P}{bb_1}\gamma\alpha\eta + \frac{P}{cc_1}\alpha\beta\zeta = 0,$$

or, what is the same thing,

$$\frac{x}{P} + \frac{\eta}{bb_1\beta} - \frac{\zeta}{cc_1\gamma} = 0:$$

and in like manner, by the elimination of (z, x) and (x, y) respectively,

$$-\frac{\xi}{aa_1\alpha} + \frac{y}{Q} + \frac{\zeta}{cc_1\gamma} = 0,$$

$$\frac{\xi}{aa_1\alpha} - \frac{\eta}{bb_1\beta} + \frac{z}{R} = 0,$$

which are the required three equations.

We have next to show that the line is a bitangent—write for a moment $aa_1\alpha$, $bb_1\beta$, $cc_1\gamma = \lambda$, μ, ν, then the three equations give

$$x,\ y,\ z = P\left(-\frac{\eta}{\mu}+\frac{\zeta}{\nu}\right),\quad Q\left(-\frac{\zeta}{\nu}+\frac{\xi}{\lambda}\right),\quad R\left(-\frac{\xi}{\lambda}+\frac{\eta}{\mu}\right),$$

and we thence have

$$\begin{aligned}\Omega = \;& P^2\left(-\frac{\xi\eta}{\mu}+\frac{\zeta\xi}{\nu}\right)^2 \\ &+ Q^2\left(-\frac{\eta\zeta}{\nu}+\frac{\xi\eta}{\lambda}\right)^2 \\ &+ R^2\left(-\frac{\zeta\xi}{\lambda}+\frac{\eta\zeta}{\mu}\right)^2 \\ &- 2QR\left(-\frac{\eta\zeta}{\nu}+\frac{\xi\eta}{\lambda}\right)\left(-\frac{\zeta\xi}{\lambda}+\frac{\eta\zeta}{\mu}\right) \\ &- 2RP\left(-\frac{\zeta\xi}{\lambda}+\frac{\eta\zeta}{\mu}\right)\left(-\frac{\xi\eta}{\mu}+\frac{\zeta\xi}{\nu}\right) \\ &- 2PQ\left(-\frac{\xi\eta}{\mu}+\frac{\zeta\xi}{\nu}\right)\left(-\frac{\eta\zeta}{\nu}+\frac{\xi\eta}{\lambda}\right);\end{aligned}$$

which, putting further $P\lambda,\ Q\mu,\ R\nu = \mathrm{a},\ \mathrm{b},\ \mathrm{c}$, becomes

$$\begin{aligned}\Omega = \ &\frac{\eta^2\zeta^2}{\mu^2\nu^2}(\mathrm{b}+\mathrm{c})^2\\ &+\frac{\zeta^2\xi^2}{\nu^2\lambda^2}(\mathrm{c}+\mathrm{a})^2\\ &+\frac{\xi^2\eta^2}{\lambda^2\mu^2}(\mathrm{a}+\mathrm{b})^2\\ &+\frac{2\xi^2\eta\zeta}{\lambda^2\mu\nu}\{-\mathrm{a}\,(\mathrm{a}+\mathrm{b}+\mathrm{c})+\mathrm{bc}\}\\ &+\frac{2\xi\eta^2\zeta}{\lambda\mu^2\nu}\{-\mathrm{b}\,(\mathrm{a}+\mathrm{b}+\mathrm{c})+\mathrm{ca}\}\\ &+\frac{2\xi\eta\zeta^2}{\lambda\mu\nu^2}\{-\mathrm{c}\,(\mathrm{a}+\mathrm{b}+\mathrm{c})+\mathrm{ab}\}.\end{aligned}$$

But here

$$\begin{aligned}\mathrm{a} &= aa_1(bc_1-b_1c)(bc-b_1c_1) = aa_1\{(b^2+b_1^2)\,cc_1-(c^2+c_1^2)\,bb_1\},\\ \mathrm{b} &= bb_1\,(ca_1-c_1a)(ca-c_1a_1) = bb_1\,\{(c^2+c_1^2)\,aa_1-(a^2+a_1^2)\,cc_1\},\\ \mathrm{c} &= cc_1\,(ab_1-a_1b)(ab-a_1b_1) = cc_1\,\{(a^2+a_1^2)\,bb_1-(b^2+b_1^2)\,aa_1\}.\end{aligned}$$

Hence

$$\mathrm{a}+\mathrm{b}+\mathrm{c}=0,$$

and we obtain

$$\begin{aligned}\Omega &= \left(\frac{\mathrm{a}\eta\zeta}{\mu\nu}+\frac{\mathrm{b}\zeta\xi}{\nu\lambda}+\frac{\mathrm{c}\xi\eta}{\lambda\mu}\right)^2,\\ &= \left\{\frac{aa_1\alpha P}{bb_1cc_1\beta\gamma}\eta\zeta+\frac{bb_1\beta Q}{cc_1aa_1\gamma\alpha}\zeta\xi+\frac{cc_1\gamma R}{aa_1bb_1\alpha\beta}\xi\eta\right\}^2,\\ &= \frac{1}{(aa_1bb_1cc_1\alpha\beta\gamma)^2}(a^2a_1^2\alpha^2P\eta\zeta+b^2b_1^2\beta^2Q\zeta\xi+c^2c_1^2\gamma^2R\xi\eta)^2,\end{aligned}$$

a perfect square, and the line 23 is thus a bitangent.

We have next to prove the equivalence of the two equations

$$\frac{x}{bc-b_1c_1}+\frac{y}{ca-c_1a_1}+\frac{z}{ab-a_1b_1}=0,$$

and

$$\frac{\xi}{g_2h_2-g_3h_3}+\frac{\eta}{h_2f_2-h_3f_3}+\frac{\zeta}{f_2g_2-f_3g_3}=0.$$

Starting from the former equation written in the form

$$-QRx-RPy-PQz=0,$$

and combining herewith the three fundamental equations

$$\begin{aligned}a\,x+b\,y+c\,z+f\,\xi+g\,\eta+h\,\zeta&=0,\\ a_1x+b_1y+c_1z+f_1\xi+g_1\eta+h_1\zeta&=0,\\ a_2x+b_2y+c_2z+f_2\xi+g_2\eta+h_2\zeta&=0,\end{aligned}$$

if we eliminate from these the (x, y, z), we obtain the following equation in (ξ, η, ζ)

$$\begin{vmatrix} a, & b, & c, & f\xi+g\eta+h\zeta \\ a_1, & b_1, & c_1, & f_1\xi+g_1\eta+h_1\zeta \\ a_2, & b_2, & c_2, & f_2\xi+g_2\eta+h_2\zeta \\ -QR, & -RP, & -PQ & \end{vmatrix}=0,$$

which, observing that the values of $-QR$, $-RP$, $-PQ$ are

$$-a^2bc-a_1^2b_1c_1+aa_1(bc_1+b_1c),\quad -ab^2c-a_1b_1^2c_1+bb_1(ca_1+c_1a),\quad -abc^2-a_1b_1c_1^2+cc_1(ab_1+a_1b),$$

is immediately transformed into

$$\begin{vmatrix} a, & b, & c, & f\xi+g\eta+h\zeta \\ a_1, & b_1, & c_1, & f_1\xi+g_1\eta+h_1\zeta \\ a_2, & b_2, & c_2, & f_2\xi+g_2\eta+h_2\zeta \\ aa_1(bc_1+b_1c), & bb_1(ca_1+c_1a), & cc_1(ab_1+a_1b), & \begin{cases} abc\,(f\xi+g\eta+h\zeta) \\ +\,a_1b_1c_1(f_1\xi+g_1\eta+h_1\zeta) \end{cases} \end{vmatrix}=0$$

Here the coefficient of ξ is

$$\begin{vmatrix} a, & b, & c, & f \\ a_1, & b_1, & c_1, & f_1 \\ a_2, & b_2, & c_2, & f_2 \\ aa_1(bc_1+b_1c), & bb_1(ca_1+c_1a), & cc_1(ab_1+a_1b), & abcf+a_1b_1c_1f_1 \end{vmatrix}$$

which is of the form $lf+l_1f_1+l_2f_2$; and we find without difficulty

$$\begin{aligned} l = {} & a\,a_2(bc-b_1c_1)(bc_1-b_1c), && = a\,a_2P\alpha \\ & +b\,b_2\,(ca-c_1a_1)(ca_1-c_1a) && +b\,b_2Q\beta \\ & +c\,c_2\,(ab-a_1b_1)(ab_1-a_1b) && +c\,c_2R\gamma; \\ l_1 = {} & -a_1a_2(bc-b_1c_1)(bc_1-b_1c), && =-a_1a_2P\alpha \\ & -b_1b_2\,(ca-c_1a_1)(ca_1-c_1a) && -b_1b_2Q\beta \\ & -c_1c_2\,(ab-a_1b_1)(ab_1-a_1b) && -c_1c_2R\gamma; \end{aligned}$$

$$l_2=-(bc_1-b_1c)(ca_1-c_1a)(ab_1-a_1b)=-\alpha\beta\gamma.$$

Hence

$$lf+l_1f_1+l_2f_2=(af-a_1f_1)\,a_2P\alpha+(bf-b_1f_1)\,b_2Q\beta+(cf-c_1f_1)\,c_2R\gamma-\alpha\beta\gamma f_2,$$

which, observing that

$$af-a_1f_1=0,\quad bf-b_1f_1=-\frac{\gamma}{aa_1},\quad cf-c_1f_1=\frac{\beta}{aa_1},$$

becomes

$$=-\,b_2Q\frac{\beta\gamma}{aa_1}+c_2R\frac{\beta\gamma}{aa_1}-\alpha\beta\gamma f_2,$$

or, in the last term writing $f_2 = \frac{1}{a_2}$, this is

$$= \frac{\beta\gamma}{aa_1a_2}\{-a_2b_2Q + a_2c_2R - aa_1\alpha\},$$

$$= \frac{\beta\gamma}{aa_1a_2}\{-a_2b_2(ca - c_1a_1) + a_2c_2(ab - a_1b_1) - aa_1(bc_1 - b_1c)\},$$

$$= \frac{\beta\gamma}{aa_1a_2}\begin{vmatrix} ab, & ac, & 1 \\ a_1b_1, & a_1c_1, & 1 \\ a_2b_2, & a_2c_2, & 1 \end{vmatrix}, \quad = \beta\gamma \begin{vmatrix} b, & c, & f \\ b_1, & c_1, & f_1 \\ b_2, & c_2, & f_2 \end{vmatrix};$$

viz. representing for shortness the last-mentioned determinant by (bcf), the coefficient of ξ is $= \beta\gamma(bcf)$. Similarly the coefficients of η, ζ are equal $\gamma\alpha(cag)$, and $\alpha\beta(abh)$, respectively. Hence, for convenience dividing by $\alpha\beta\gamma$, the original equation

$$\frac{x}{bc - b_1c_1} + \frac{y}{ca - c_1a_1} + \frac{z}{ab - a_1b_1} = 0,$$

expressed in terms of (ξ, η, ζ), becomes

$$\frac{1}{\alpha}(bcf)\,\xi + \frac{1}{\beta}(cag)\,\eta + \frac{1}{\gamma}(abh)\,\zeta = 0;$$

and it is to be shown that this is, in fact, equivalent to

$$\frac{\xi}{g_2h_2 - g_3h_3} + \frac{\eta}{h_2f_2 - h_3f_3} + \frac{\zeta}{f_2g_2 - f_3g_3} = 0.$$

We ought therefore to have

$$(g_2h_2 - g_3h_3)\frac{1}{\alpha}(bcf) = (h_2f_2 - h_3f_3)\frac{1}{\beta}(cag) = (f_2g_2 - f_3g_3)\frac{1}{\gamma}(abh),$$

and it will be sufficient to prove the first of these equalities, say

$$(g_2h_2 - g_3h_3)\,\beta\,(bcf) = (h_2f_2 - h_3f_3)\,\alpha\,(cag).$$

Multiplying by c_2c_3, this becomes

$$c_3[g_2\beta(bcf) - f_2\alpha(cag)]$$
$$- c_2g_3\beta(bcf) + c_2f_3\alpha(cag) = 0,$$

viz. this is

$$(\lambda c + \lambda_1c_1 + \lambda_2c_2)[g_2\beta(bcf) - f_2\alpha(cag)]$$
$$- (\lambda g + \lambda_1g_1 + \lambda_2g_2)\,c_2\beta(bcf)$$
$$+ (\lambda f + \lambda_1f_1 + \lambda_2f_2)\,c_2\alpha(cag) = 0.$$

The term in λ_2 disappears, and the equation is

$$\begin{aligned}&\lambda\ [(c\,g_2-c_2g)\,\beta\,(bcf)-(c\,f_2-c_2f)\,\alpha\,(cag)]\\ &+\lambda_1\,[(c_1g_2-c_2g_1)\,\beta\,(bcf)-(c_1f_2-c_2f_1)\,\alpha\,(cag)]=0.\end{aligned}$$

But we have $\lambda : \lambda_1 = \frac{1}{\Lambda} : \frac{1}{\Lambda_1}$, that is, $\Lambda\lambda - \Lambda_1\lambda_1 = 0$, or if we write for Λ and Λ_1 their values $(1F_1F_2)$, $(=-(1F_2F_1))$, and $(1F_2F)$ respectively, then the equation connecting λ, λ_1 is

$$\lambda\,(1F_2F_1)+\lambda_1\,(1F_2F)=0,$$

where $(1F_2F_1)$ and $(1F_2F)$ denote the determinants

$$\begin{vmatrix}1, & af_1+a_1f, & af_2+a_2f\\ 1, & bg_1+b_1g, & bg_2+b_2g\\ 1, & ch_1+c_1h, & ch_2+c_2h\end{vmatrix},\quad \begin{vmatrix}1, & af_1+a_1f, & a_1f_2+a_2f_1\\ 1, & bg_1+b_1g, & b_1g_2+b_2g_1\\ 1, & ch_1+c_1h, & c_1h_2+c_2h_1\end{vmatrix}.$$

Multiplying by cc_1c_2, the equation may be written

$$\lambda\begin{vmatrix}af, & af_1+a_1f, & af_2+a_2f\\ bg, & bg_1+b_1g, & bg_2+b_2g\\ cc_1c_2, & c_2(c^2+c_1^2), & c_1(c^2+c_2^2)\end{vmatrix}+\lambda_1\begin{vmatrix}a_1f_1, & af_1+a_1f, & a_1f_2+a_2f_1\\ b_1g_1, & bg_1+b_1g, & b_1g_2+b_2g_1\\ cc_1c_2, & c_2(c^2+c_1^2), & c_1(c^2+c_2^2)\end{vmatrix}=0.$$

This should agree with the equation in (λ, λ_1) obtained above: and it can, in fact, be shown that the coefficients

$$(c\,g_2-c_2g)\,\beta\,(bcf)-(c\,f_2-c_2f)\,\alpha\,(cag),$$

and

$$(c_1g_2-c_2g_1)\,\beta\,(bcf)-(c_1f_2-c_2f_1)\,\alpha\,(cag),$$

are equal to the two determinants respectively; it will be sufficient to prove one of the two relations, say

$$-(cg_2-c_2g)\,(ca_1-c_1a)\,(bcf)+(cf_2-c_2f)\,(bc_1-b_1c)\,(cag)$$
$$+\begin{vmatrix}af, & af_1+a_1f, & af_2+a_2f\\ bg, & bg_1+b_1g, & bg_2+b_2g\\ cc_1c_2, & c_2(c^2+c_1^2), & c_1(c^2+c_2^2)\end{vmatrix}=0;$$

where it will be recollected that (bcf) and (cag) stand for the determinants

$$\begin{vmatrix}b, & c, & f\\ b_1, & c_1, & f_1\\ b_2, & c_2, & f_2\end{vmatrix},\quad \begin{vmatrix}c, & a, & g\\ c_1, & a_1, & g_1\\ c_2, & a_2, & g_2\end{vmatrix},$$

respectively: each term is thus of the seventh order in all the letters conjointly, but is quadric in f, f_1, f_2, g, g_1, g_2. Instead of α, β, γ used hitherto to denote bc_1-b_1c,

$ca_1 - c_1 a$, $ab_1 - a_1 b$, it will be convenient to call these α_2, β_2, γ_2, and to consider the complete system of coefficients

$$\alpha,\ \alpha_1,\ \alpha_2 = b_1 c_2 - b_2 c_1,\ b_2 c - b c_2,\ b c_1 - b_1 c;\quad \beta,\ \beta_1,\ \beta_2 = c_1 a_2 - c_2 a_1,\ c_2 a - c a_2,\ c a_1 - c_1 a;$$
$$\gamma,\ \gamma_1,\ \gamma_2 = a_1 b_2 - a_2 b_1,\ a_2 b - a b_2,\ a b_1 - a_1 b;$$

and to denote by ∇ the determinant formed with α, β, γ; $\alpha_1, \beta_1, \gamma_1$; $\alpha_2, \beta_2, \gamma_2$. If we then expand in terms of the products $(f, f_1, f_2)(g, g_1, g_2)$, we find first

$$\begin{aligned}
&-(cg_2 - c_2 g)(ca_1 - c_1 a)(bcf) + (cf_2 - c_2 f)(bc_1 - b_1 c)(cag)\\
&= -(cg_2 - c_2 g)\,\beta_2(\alpha f + \alpha_1 f_1 + \alpha_2 f_2) + (cf_2 - c_2 f)\,\alpha_2(\beta g + \beta_1 g_1 + \beta_2 g_2),\\
&= -\alpha_2\beta + \alpha\beta_2 (= -c_1\nabla)\, c_2 fg\\
&\quad - \alpha_2\beta_1 c_2 f g_1\\
&\quad + \alpha_1\beta_2 c_2 f_1 g\\
&\quad - \beta_2(\ \ \alpha c + \alpha_2 c_2 = -\alpha_1 c_1) f g_2\\
&\quad - \alpha_2(-\beta c - \beta_2 c_2 = \ \ \beta_1 c_1) f_2 g\\
&\quad - \alpha_1\beta_2 c f_1 g_2\\
&\quad + \alpha_2\beta_1 c f_2 g_1;
\end{aligned}$$

and next

$$\begin{vmatrix} af, & af_1 + a_1 f, & af_2 + a_2 f\\ bg, & bg_1 + b_1 g, & bg_2 + b_2 g\\ cc_1 c_2, & c_2(c^2 + c_1^2), & c_1(c^2 + c_2^2)\end{vmatrix}$$

$$\begin{aligned}
&= cc_1 c_2\gamma + c_2(c^2 + c_1^2)\gamma_1 + c_1(c^2 + c_2^2)\gamma_2 (= c_1 c_2\nabla + c^2(c_1\gamma_2 + c_2\gamma_1)) fg\\
&\quad + c_1(\beta_1 c_2 b + c^2 ab) f g_1\\
&\quad + c_1(\alpha_1 c_2 a - c^2 ab) f_1 g\\
&\quad + c_2(\beta_2 c_1 b - c^2 ab) f g_2\\
&\quad + c_2(\alpha_2 c_1 a + c^2 ab) f_2 g\\
&\quad + c_1 c_2 abc f_1 g_2\\
&\quad - c_1 c_2 abc f_2 g_1.
\end{aligned}$$

Uniting the two parts, we ought to have

$$\begin{aligned}
0 &= c^2(c_2\gamma_1 + c_1\gamma_2) fg\\
&\quad + [\beta_1 c_2(c_1 b - \alpha_2) + c_1 c^2 ab] f g_1\\
&\quad + [\alpha_1 c_2(c_1 a + \beta_2) - c_1 c^2 ab] f_1 g\\
&\quad + [c_1\beta_2(c_2 b + \alpha_1) - c_2 c^2 ab] f g_2\\
&\quad + [c_1\alpha_2(c_2 a - \beta_1) + c_2 c^2 ab] f_2 g\\
&\quad + c(c_1 c_2 ab - \alpha_1\beta_2) f_1 g_2\\
&\quad - c(c_1 c_2 ab - \alpha_2\beta_1) f_2 g_1.
\end{aligned}$$

Substituting for α_1, β_1, γ_1, α_2, β_2, γ_2, then after a slight reduction it is found that the whole equation divides by c, and omitting this factor, we have

$$\begin{aligned}
0 = {} & c\,[(c_2a_2 - c_1a_1)\,b - (c_2b_2 - c_1b_1)\,a]\,fg \\
& + [\ \ cc_1ab + b_1c_2\,(c_2a - ca_2)\ \ ]\,fg_1 \\
& + [- cc_1ab + a_1c_2\,(b_2c - bc_2)\ \ ]\,f_1g \\
& + [- cc_2ab + c_1b_2\,(ca_1 - c_1a)\ \ ]\,fg_2 \\
& + [\ \ cc_2ab + c_1a_2\,(bc_1 - b_1c)\ \ ]\,f_2g \\
& + [- c^2a_1b_2 + ca\,b_2c_1 + bc\,a_1c_2\ \ ]\,f_1g_2 \\
& + [\ \ c^2a_2b_1 - bc\,a_2c_1 - ca\,b_1c_2\ \ ]\,f_2g_1.
\end{aligned}$$

Writing here $af = bg = \ldots = 1$, the equation becomes

$$\begin{aligned}
0 = {} & c\,(c_2a_2 - c_1a_1)\,f - c\,(c_2b_2 - c_1b_1)\,g \\
& + cc_1bg_1 + c_2\,(c_2a - ca_2)\,f \\
& - cc_1af_1 + c_2\,(b_2c - bc_2)\,g \\
& - cc_2bg_2 + c_1\,(ca_1 - c_1a)\,f \\
& + cc_2af_2 + c_1\,(bc_1 - b_1c)\,g \\
& - c^2 + acc_1f_1 + bcc_2g_2 \\
& + c^2 - bcc_1g_1 - cac_2f_2,
\end{aligned}$$

where all the terms in f_1, g_1 destroy each other; the equation thus becomes

$$\begin{aligned}
0 = {} & (\ \ cc_2a_2 - cc_1a_1 + c_2^2a - cc_2a_2 + cc_1a_1 - c_1^2a)\,f \\
& + (- cc_2b_2 + cc_1b_1 + cc_2b_2 - c_2^2b + c_1^2b - cc_1b_1)\,g
\end{aligned}$$

that is,

$$0 = (c_2^2 - c_1^2)\,af + (- c_2^2 + c_1^2)\,bg.$$

Or again writing $af = bg = 1$, we have the identity $0 = 0$. This completes the proof of the equivalence of the two equations

$$\frac{x}{bc - b_1c_1} + \frac{y}{ca - c_1a_1} + \frac{z}{ab - a_1b_1} = 0, \quad \frac{\xi}{g_2h_2 - g_3h_3} + \frac{\eta}{h_2f_2 - h_3f_3} + \frac{\zeta}{f_2g_2 - f_3g_3} = 0.$$

We have obviously three new forms derived from the equation in (ξ, η, ζ) in like manner as we had previously three new forms derived from the equation in (x, y, z); the equivalence of the 8 forms to each other is thus shown, and the proof is now complete for each of the 28 bitangents.

Cambridge, 15 *Dec.* 1882.

In what follows, instead of the four equations of the form

$$ax+by+cz+f\xi+g\eta+h\zeta=0,$$

I take the first equation in Riemann's form with the coefficients unity. The equations thus are

$$\begin{gathered} x+\ y+\ z+\ \xi+\ \eta+\ \zeta=0,\\ a\,x+b\,y+c\,z+f\,\xi+g\,\eta+h\,\zeta=0,\\ a_1x+b_1y+c_1z+f_1\xi+g_1\eta+h_1\zeta=0,\\ a_2x+b_2y+c_2z+f_2\xi+g_2\eta+h_2\zeta=0, \end{gathered}$$

and if we assume

$$a_2=\lambda+\lambda_1a+\lambda_2a_1,\ \&\text{c.}$$

then we have

$$\begin{aligned} 1-\lambda^2-\lambda_1^2-\lambda_2^2&=\lambda_1\lambda_2(af_1+a_1f)+\lambda_2\lambda(a_1+f_1)+\lambda\lambda_1(a+f),\\ \text{,,}\quad&=\lambda_1\lambda_2(bg_1+b_1g)+\lambda_2\lambda(b_1+g_1)+\lambda\lambda_1(b+g),\\ \text{,,}\quad&=\lambda_1\lambda_2(ch_1+c_1h)+\lambda_2\lambda(c_1+h_1)+\lambda\lambda_1(c+h), \end{aligned}$$

and consequently

$$\lambda_1\lambda_2\ :\ \lambda_2\lambda\ :\ \lambda\lambda_1=P\ :\ Q\ :\ R,$$

or say

$$\lambda\ :\ \lambda_1\ :\ \lambda_2=QR\ :\ RP\ :\ PQ,$$

where

$$P,\ Q,\ R=\begin{vmatrix}1, & a_1+f_1, & a+f\\ 1, & b_1+g_1, & b+g\\ 1, & c_1+h_1, & c+h\end{vmatrix},\quad \begin{vmatrix}1, & a+f, & af_1+a_1f\\ 1, & b+g, & bg_1+b_1g\\ 1, & c+h, & ch_1+c_1h\end{vmatrix},\quad \begin{vmatrix}1, & af_1+a_1f, & a_1+f_1\\ 1, & bg_1+b_1g, & b_1+g_1\\ 1, & ch_1+c_1h, & c_1+h_1\end{vmatrix}.$$

Suppose that one of these determinants, to fix the ideas, say Q, is $=0$; we have $\lambda\ :\ \lambda_1\ :\ \lambda_2=0\ :\ RP\ :\ 0$; that is, λ and λ_2 each $=0$; and consequently $\lambda_1=1$: that is, we have $a_2, b_2, c_2, f_2, g_2, h_2=a, b, c, f, g, h$, the fourth equation identical with the second, or say the second equation,

$$ax+by+cz+f\xi+g\eta+h\zeta=0,$$

is a double equation: as just seen, the condition for this is

$$Q=\begin{vmatrix}1, & a+f, & af_1+a_1f\\ 1, & b+g, & bg_1+b_1g\\ 1, & c+h, & ch_1+c_1h\end{vmatrix}=0,$$

say this is

$$L(af_1+a_1f)+M(bg_1+b_1g)+N(ch_1+c_1h)=0,$$

if for shortness

$$L,\ M,\ N=b+g-c-h,\quad c+h-a-f,\quad a+f-b-g.$$

A special solution is if $a_1, b_1, c_1 = a^2, b^2, c^2$; for then

$$a_1,\ b_1,\ c_1,\ f_1,\ g_1,\ h_1 = a^2,\ b^2,\ c^2,\ f^2,\ g^2,\ h^2;$$

consequently $af_1 + a_1 f,\ bg_1 + b_1 g,\ ch_1 + c_1 h = a + f,\ b + g,\ c + h$: and the condition in question $Q = 0$ is thus satisfied.

It is to be shown that, in the general case of the equation $Q = 0$, the curve is a nodal quartic having the node at the point

$$x : y : z : \xi : \eta : \zeta = fL : gM : hN : aL : bM : cN,$$

and that in the special case $a_1, b_1, c_1 = a^2, b^2, c^2$ the node is a fleflecnode, viz. a node which is a point of inflexion on each of its two branches. In the former case, the six tangents from the node are

$$\begin{aligned}
ax + by + cz &= 0 \quad \text{or} \quad f\xi + g\eta + h\zeta = 0,\\
f\xi + by + cz &= 0 \quad ,, \quad ax + g\eta + h\zeta = 0,\\
ax + g\eta + cz &= 0 \quad ,, \quad f\xi + by + h\zeta = 0,\\
ax + by + h\zeta &= 0 \quad ,, \quad f\xi + g\eta + cz = 0,
\end{aligned}$$

$$\frac{x}{1-bc} + \frac{y}{1-ca} + \frac{z}{1-ab} = 0, \quad \text{or} \quad \frac{\xi}{gh - g_1h_1} + \frac{\eta}{hf - h_1f_1} + \frac{\zeta}{fg - f_1g_1} = 0,$$

$$\frac{x}{bc - b_1c_1} + \frac{y}{ca - c_1a_1} + \frac{z}{ab - a_1b_1} = 0, \quad ,, \quad \frac{\xi}{1-gh} + \frac{\eta}{1-hf} + \frac{\zeta}{1-fg} = 0.$$

In the latter case, the same equations represent the four tangents from the node and the two tangents at the node respectively.

For the proof, we assume that the four lines

$$\begin{aligned}
ax + by + cz &= 0,\\
f\xi + by + cz &= 0,\\
ax + g\eta + cz &= 0,\\
ax + by + h\zeta &= 0,
\end{aligned}$$

have a common intersection: this gives $ax = f\xi,\ by = g\eta,\ cz = h\zeta$; or say

$$x,\ y,\ z,\ \xi,\ \eta,\ \zeta = fL,\ gM,\ hN,\ aL,\ bM,\ cN,$$

where for the moment L, M, N are indeterminate quantities: but substituting these values in the first, second and third equations, we find

$$\begin{aligned}
(a+f)\,L + \quad (b+g)\,M + \quad (c+h)\,N &= 0,\\
L + \qquad\quad M + \qquad\quad N &= 0,\\
(af_1 + a_1f)\,L + (bg_1 + b_1g)\,M + (ch_1 + c_1h)\,N &= 0,
\end{aligned}$$

giving for L, M, N the values $b + g - c - h,\ c + h - a - f,\ a + f - b - g$ already attributed to these letters.

And we see further that the point in question

$$x,\ y,\ z,\ \xi,\ \eta,\ \zeta, = fL,\ gM,\ hN,\ aL,\ bM,\ cN,$$

is a point on each of the lines

$$\frac{x}{1-bc} + \frac{y}{1-ca} + \frac{z}{1-ab} = 0,$$

$$\frac{x}{bc-b_1c_1} + \frac{y}{ca-c_1a_1} + \frac{z}{ab-a_1b_1} = 0.$$

In fact, for the first of these lines, we have only to verify the equation

$$fL\,(1-ca)\,(1-ab) + gM\,(1-ab)\,(1-bc) + hN\,(1-bc)\,(1-ca) = 0,$$

that is,

$$L\,(f-b-c+abc) + M\,(g-c-a+abc) + N\,(h-a-b+abc) = 0,$$

viz. this is

$$L\,(a+f) + M\,(b+g) + N\,(c+h) + (-a-b-c+abc)\,(L+M+N) = 0,$$

which is right. And similarly, for the second line, we have to verify that

$$fL\,(ca-c_1a_1)\,(ab-a_1b_1) + gM\,(ab-a_1b_1)\,(bc-b_1c_1) + hN\,(bc-b_1c_1)\,(ca-c_1a_1) = 0,$$

that is,

$$L\,(abc - bc_1a_1 - ca_1b_1 + fa_1{}^2b_1c_1) + \&c. = 0,$$

or, multiplying by $f_1g_1h_1$, this is

$$L\,(abcf_1g_1h_1 - bg_1 - ch_1 + a_1f) + M\,(abcf_1g_1h_1 - ch_1 - af_1 + b_1g) + N\,(abcf_1g_1h_1 - af_1 - bg_1 + c_1h) = 0,$$

viz. this is

$$(abcf_1g_1h_1 - af_1 - bg_1 - ch_1)\,(L+M+N) + (af_1 + a_1f)\,L + (bg_1 + b_1g)\,M + (ch_1 + c_1h)\,N = 0.$$

We have thus six of the double tangents meeting at the point

$$x,\ y,\ z,\ \xi,\ \eta,\ \zeta, = fL,\ gM,\ hN,\ aL,\ bM,\ cN,$$

which implies that this point is a node on the curve: and we at once see that the values satisfy the equation $\sqrt{x\xi} + \sqrt{y\eta} + \sqrt{z\zeta} = 0$ of the curve; viz. the equation for the values in question becomes $\sqrt{L^2} + \sqrt{M^2} + \sqrt{N^2} = 0$, and the rationalised equation is satisfied as containing the factor $L + M + N$ which is $= 0$.

But to show *à posteriori* with greater distinctness that the point is, in fact, a node, observe that, the rationalised equation being

$$\Omega = x^2\xi^2 + y^2\eta^2 + z^2\zeta^2 - 2yz\eta\zeta - 2zx\zeta\xi - 2xy\xi\eta,$$

the differential $d\Omega$ is

$$= (xd\xi + \xi dx)\,(x\xi - y\eta - z\zeta) + (yd\eta + \eta dy)\,(-x\xi + y\eta - z\zeta) + (zd\zeta + \zeta dz)\,(-x\xi - y\eta + z\zeta),$$

which for the values in question becomes

$$=L\,(f d\xi+a dx)(L^2-M^2-N^2)+M\,(g d\eta+b dy)(-L^2+M^2-N^2)+N\,(h d\zeta+c dz)(-L^2-M^2+N^2);$$

and this, in virtue of $L+M+N=0$, and therefore $L^2-M^2-N^2=2MN$, &c., is

$$=2LMN\,(a\,dx+b\,dy+c\,dz+f\,d\xi+g\,d\eta+h\,d\zeta),$$

which is $=0$ in virtue of the second equation.

Consider now in the special case $a_1,\ b_1,\ c_1=a^2,\ b^2,\ c^2$, the line

$$\frac{x}{1-bc}+\frac{y}{1-ca}+\frac{z}{1-ab}=0;$$

combining this with the first and second equations

$$x+\ y+\ z+\ \xi+\ \eta+\ \zeta=0,$$
$$ax+by+cz+f\xi+g\eta+h\zeta=0,$$

we deduce

$$x=-F\left(\frac{\eta}{b\beta}-\frac{\zeta}{c\gamma}\right),$$

$$y=-G\left(\frac{\zeta}{c\gamma}-\frac{\xi}{a\alpha}\right),$$

$$z=-H\left(\frac{\xi}{a\alpha}-\frac{\eta}{b\beta}\right),$$

if for shortness $F,\ G,\ H,\ \alpha,\ \beta,\ \gamma=bc-1,\ ca-1,\ ab-1,\ b-c,\ c-a,\ a-b$. Observe that we have $L=b+g-c-h=(b-c)\left(1-\dfrac{1}{bc}\right)$, that is, $bcL=\alpha F$; and similarly $caM=\beta G$, and $abN=\gamma H$. And if for a moment we write further

$$\xi,\ \eta,\ \zeta,\ I,\ J,\ K,=a\alpha X,\ b\beta Y,\ c\gamma Z,\ a\alpha F,\ b\beta G,\ c\gamma H,$$

values which give

$$I+J+K=(abc-a)(b-c)+(abc-b)(c-a)+(abc-c)(a-b),\quad=0,$$

then the equation $\sqrt{x\xi}+\sqrt{y\eta}+\sqrt{z\zeta}=0$ of the curve becomes

$$\sqrt{IX\,(Y-Z)}+\sqrt{JY\,(Z-X)}+\sqrt{KZ\,(X-Y)}=0,$$

which in the rationalised form is

$$I^2X^2(Y-Z)^2+J^2Y^2(Z-X)^2+K^2Z^2(X-Y)^2$$
$$-2JKYZ\,(Z-X)(X-Y)-2KIZX\,(X-Y)(Y-Z)-2IJXY\,(Y-Z)(Z-X)=0,$$

viz. this is

$$Y^2Z^2(J^2+2JK+K^2)+\ldots+2YZ\,(JK-I^2-IJ-IK)+\ldots=0,$$

which, in virtue of $I+J+K=0$, becomes

$$(IYZ+JZX+KXY)^2=0,$$

that is, the quartic is met by the line in question at the intersections (each taken twice) of the line with the conic

$$IYZ + JZX + KXY = 0,$$

that is,

$$Fa^2\alpha^2 . \eta\zeta + Gb^2\beta^2 . \zeta\xi + Hc^2\gamma^2 . \xi\eta = 0.$$

But the equation of the line in terms of ξ, η, ζ is

$$\frac{\xi}{gh - g_1h_1} + \frac{\eta}{hf - h_1f_1} + \frac{\zeta}{fg - f_1g_1} = 0,$$

that is,

$$\frac{\xi}{gh - g^2h^2} + \frac{\eta}{hf - h^2f^2} + \frac{\zeta}{fg - f^2g^2} = 0,$$

or, what is the same thing,

$$\frac{b^2c^2\xi}{F} + \frac{c^2a^2\eta}{G} + \frac{a^2b^2\zeta}{H} = 0,$$

and this is clearly a tangent to the conic; viz. the rationalised form of

$$\sqrt{Fa^2\alpha^2 . \frac{b^2c^2}{F}} + \sqrt{Gb^2\beta^2 . \frac{c^2a^2}{G}} + \sqrt{Hc^2\gamma^2 . \frac{a^2b^2}{H}} = 0,$$

that is, $\sqrt{\alpha^2} + \sqrt{\beta^2} + \sqrt{\gamma^2} = 0$ is satisfied in virtue of $\alpha + \beta + \gamma = 0$. Hence the four intersections coincide together, or the line has with the curve a quadruple intersection at the node, that is, it is a tangent at the node to a branch having an inflexion. And similarly the other line

$$\frac{x}{bc - b^2c^2} + \frac{y}{ca - c^2a^2} + \frac{z}{ab - a^2b^2} = 0, \text{ that is, } \frac{\xi}{1 - gh} + \frac{\eta}{1 - hf} + \frac{\zeta}{1 - fg} = 0,$$

is a tangent at the node to the other branch having also an inflexion; thus the node is a fleflecnode, and the lines are the tangents to the two branches.

I continue the investigation of the special case $a_1, b_1, c_1, = a^2, b^2, c^2$. The three equations give

$$\begin{aligned}
-(h - f)(f - g)\,\xi + (a - g)(a - h)\,x + (b - g)(b - h)\,y + (c - g)(c - h)\,z &= 0,\\
-(f - g)(g - h)\,\eta + (a - h)(a - f)\,x + (b - h)(b - f)\,y + (c - h)(c - f)\,z &= 0,\\
-(g - h)(h - f)\,\zeta + (a - f)(a - g)\,x + (b - f)(b - g)\,y + (c - f)(c - g)\,z &= 0;
\end{aligned}$$

writing

A, B, C, F, G, H; $\alpha, \beta, \gamma, = a^2 - 1, b^2 - 1, c^2 - 1, bc - 1, ca - 1, ab - 1$; $b - c, c - a, a - b$,

and also

$$u = \frac{1}{\alpha\beta\gamma}(Ax + By + Cz),$$

then we find

$$\begin{aligned}
\xi &= a^2(\alpha Fu - x),\\
\eta &= b^2(\beta Gu - y),\\
\zeta &= c^2(\gamma Hu - z),
\end{aligned}$$

say for shortness these values are

$$\xi,\ \eta,\ \zeta = pu - a^2x,\quad qu - b^2y,\quad ru - c^2z,$$

where

$$p,\ q,\ r = a^2\alpha F,\ b^2\beta G,\ c^2\gamma H.$$

We have moreover $f^2\beta\gamma\,.\,\xi = \beta\gamma(\alpha Fu - x),\ = F(Ax + By + Cz) - \beta\gamma x$; or observing that $GH - AF = -\beta\gamma$, &c., we have

$$\begin{aligned} f^2\beta\gamma\xi &= GHx + BFy + CFz,\\ g^2\gamma\alpha\eta &= AGx + HFy + CGz,\\ h^2\alpha\beta\zeta &= AHx + BHy + FGz. \end{aligned}$$

It is to be shown that the tangents from the fleflecnode

$$\begin{aligned} ax + by + cz &= 0,\\ f\xi + by + cz &= 0,\\ ax + g\eta + cz &= 0,\\ ax + by + h\zeta &= 0, \end{aligned}$$

touch the quartic at its intersections with the line $u = 0$.

To prove this, for the first line we have $ax + by + cz = 0$, $f\xi + g\eta + h\zeta = 0$, and consequently $x\xi - y\eta - z\zeta = bhy\zeta + cgz\eta$. But the equation of the curve is

$$\Omega = (x\xi - y\eta - z\zeta)^2 - 4yz\eta\zeta = 0\,;$$

and this becomes thus

$$\Omega = (bhy\zeta + cgz\eta)^2 - 4bcghyz\eta\zeta = (bhy\zeta - cgz\eta)^2 = 0\,;$$

and we thus find that, for the four lines respectively, the equations for the points of contact are

$$\begin{aligned} &ax + by + cz = 0, && bhy\zeta - cgz\eta = 0,\\ &f\xi + by + cz = 0, && \text{,,}\\ &ax + g\eta + cz = 0, && gh\eta\zeta - bcyz = 0,\\ &ax + by + h\zeta = 0, && \text{,,}\quad . \end{aligned}$$

But for the first line we have

$$\begin{aligned} bhy\zeta - cgz\eta &= bhyc^2(\gamma Hu - z) - cgzb^2(\beta Gu - y),\\ &= bcy(\gamma Hu - z) - bcz(\beta Gu - y),\\ &= bcu(\gamma Hy - \beta Gz), \end{aligned}$$

so that the intersections with the conic are given by the equations $\gamma Hy - \beta Gz = 0$, and $u = 0$ respectively: the first of these corresponds to the fleflecnode (in fact, we have $\gamma H\,.\,gM - \beta G\,.\,hN = 0$, viz. multiplying by abc, this is

$$\gamma H\,.\,caM - \beta G\,.\,abN = 0,$$

which is right), hence the second corresponds to the point of contact, that is, the point of contact lies on the line $u = 0$. And similarly, for each of the other three tangents, the point of contact lies on the same line $u = 0$.

It is clear that writing $T_1=0$, $T_2=0$, $T_3=0$, $T_4=0$ for the four tangents from the fleflecnode and $T_5=0$, $T_6=0$ for the tangents at this point, the equation of the curve must be expressible in the form

$$\lambda u^2 T_5 T_6 + \mu T_1 T_2 T_3 T_4 = 0.$$

The reduction to this form is, I think, effected by means of the formulæ which give ξ, η, ζ in terms of x, y, z, more readily than by means of simpler formulæ for ξ, η, ζ in terms of x, y, z and u. We have

$$\Omega = x^2\xi^2 + y^2\eta^2 + z^2\zeta^2 - 2yz\eta\zeta - 2zx\zeta\xi - 2xy\xi\eta = 0,$$

and hence

$$\begin{aligned}\Omega' = f^4g^4h^4\alpha^2\beta^2\gamma^2\Omega = \;& g^4h^4\alpha^2x^2\ (GHx + BFy + CFz)^2\\ &+ h^4f^4\beta^2y^2(AGx + HFy + CGz)^2\\ &+ f^4g^4\gamma^2z^2\ (AHx + BHy + FGz)^2\\ &- 2f^4g^2h^2\beta\gamma yz\,(AGx + HFy + CGz)(AHx + BHy + FGz)\\ &- 2g^4h^2f^2\gamma\alpha zx\,(AHx + BHy + FGz)(GHx + BFy + CFz)\\ &- 2h^4f^2g^2\alpha\beta xy(GHx + BFy + CFz)(AGx + HFy + CGz) = 0.\end{aligned}$$

The result after some reductions, and putting for shortness

$$\begin{aligned}h^2\beta CG - g^2\gamma BH &= \mathfrak{A},\\ f^2\gamma AH - h^2\alpha CF &= \mathfrak{B},\\ g^2\alpha BF - f^2\beta AG &= \mathfrak{C},\end{aligned}$$

is

$$\begin{aligned}\Omega' = \;& g^4h^4G^2H^2\alpha^2x^4 + h^4f^4H^2F^2\beta^2y^4 + f^4g^4F^2G^2\gamma^2z^4\\ &+ 2h^2f^4HF\beta\,.\,\mathfrak{A}y^3z + 2g^2f^4FG\gamma\,.\,\mathfrak{A}yz^3\\ &+ 2f^2g^4FG\gamma\,.\,\mathfrak{B}z^3x + 2h^2g^4GH\alpha\,.\,\mathfrak{B}zx^3\\ &+ 2g^2h^4GH\alpha\,.\,\mathfrak{C}x^3y + 2f^2h^4HF\beta\,.\,\mathfrak{C}xy^3\\ &+ f^4y^2z^2\ (\mathfrak{A}^2 - 2g^2h^2\beta\gamma F^2GH)\\ &+ g^4z^2x^2\ (\mathfrak{B}^2 - 2h^2f^2\gamma\alpha FG^2H)\\ &+ h^4x^2y^2\ (\mathfrak{C}^2 - 2f^2g^2\alpha\beta FGH^2)\\ &+ 2g^2h^2x^2yz\ \{-\mathfrak{B}\mathfrak{C} - f^2GH\,(2f^2\beta\gamma A^2 + g^2\gamma\alpha BH + h^2\alpha\beta CG)\}\\ &+ 2h^2f^2xy^2z\,\{-\mathfrak{C}\mathfrak{A} - g^2HF\,(f^2\beta\gamma AH + 2g^2\gamma\alpha B^2 + h^2\alpha\beta CF)\}\\ &+ 2f^2g^2xyz^2\ \{-\mathfrak{A}\mathfrak{B} - h^2FG\,(f^2\beta\gamma AG + g^2\gamma\alpha BF + 2h^2\alpha\beta C^2)\} = 0.\end{aligned}$$

We have

$$u = \frac{1}{\alpha\beta\gamma}(Ax + By + Cz),$$

$$T_5 = \frac{x}{1-bc} + \frac{y}{1-ca} + \frac{z}{1-ab},$$

$$T_6 = \frac{x}{bc - b^2c^2} + \frac{y}{ca - c^2a^2} + \frac{z}{ab - a^2b^2},$$

or, as these equations may be written,

$$\begin{aligned} \alpha\beta\gamma u &= Ax + By + Cz, \\ -FGHT_5 &= GHx + HFy + FGz, \\ -abcFGHT_6 &= aGHx + bHFy + cFGz; \end{aligned}$$

whence

$$abcF^2G^2H^2 \,.\, \alpha\beta\gamma \,.\, uT_5T_6 = (Ax + By + Cz)^2 (GHx + HFy + FGz)(aGHx + bHFy + cFGz),$$

where the coefficient of x^4 is $= aA^2G^2H^2$.

Again $T_1 = ax + by + cz$. For T_2 we have

$$\begin{aligned} bc\beta\gamma T_2 &= abcf\beta\gamma (f\xi + by + cz) \\ &= abc(GHx + BFy + CFz) + bc\beta\gamma(by + cz), \end{aligned}$$

where the coefficient of x is $abcGH$. The coefficient of y is

$$\begin{aligned} abc(b^2-1)(bc-1) + b^2c(c-a)(a-b), \quad &= bc(ab^3c - b^2c - a^2b + a), \\ &= bc(ab-1)(b^2c - a), \\ &= bcH(b^2c - a), \end{aligned}$$

and similarly the coefficient of z is $= bcG(bc^2 - a)$. We have thus the expression for T_2; and forming in like manner the expressions for T_3 and T_4, we may write

$$\begin{aligned} T_1 &= ax + by + cz, \\ bc\beta\gamma T_2 &= abcGHx + bcH(b^2c - a)y + bcG(bc^2 - a)z, \\ ca\gamma\alpha T_3 &= caH(ca^2 - b)x + abcHFy + caF(c^2a - b)z, \\ ab\alpha\beta T_4 &= abG(a^2b - c)x + abF(ab^2 - c)y + abcFGz, \end{aligned}$$

whence in $a^2b^2c^2\alpha^2\beta^2\gamma^2T_1T_2T_3T_4$ the coefficient of x^4 is

$$= a^4b^2c^2 \,.\, G^2H^2 \,.\, (ca^2 - b)(a^2b - c) = a^4b^2c^2 \,.\, G^2H^2 (bcA^2 - a^2\alpha^2).$$

We hence obtain the required identity: viz. this is

$$\begin{aligned} a^6b^6c^6\Omega' = a^3b^3c^3 \,.\, abcF^2G^2H^2\alpha\beta\gamma \,.\, uT_5T_6 \\ -1 \,.\, \quad a^2b^2c^2\alpha^2\beta^2\gamma^2 \,.\, T_1T_2T_3T_4; \end{aligned}$$

for here comparing the terms in x^4, the factor G^2H^2 divides out, and omitting it, we have

$$a^6b^2c^2\alpha^2 = a^3b^3c^3 \,.\, aA^2 - a^4b^2c^2 \,.\, (bcA^2 - a^2\alpha^2),$$

which is right. If for Ω' we substitute its value, $= f^4g^4h^4\alpha^2\beta^2\gamma^2\Omega$, then the identity divides by $a^2b^2c^2\alpha\beta\gamma$, and omitting this factor, we obtain the equation of the curve in the form

$$\alpha\beta\gamma\Omega = a^2b^2c^2FGH \,.\, uT_5T_6 - \alpha\beta\gamma \,.\, T_1T_2T_3T_4 = 0;$$

the required form, putting in evidence the two tangents at the fleflecnode, and the four tangents from this point.

Cambridge, 3rd January, 1883.