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On wave guide-type propagation in elastic fibre-reinforced composites

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A SECOND-ORDER microstructure theory for elastic fibre-reinforced composites is presented for the purpose of studying plane harmonic wave propagation in the direction of the fibres. Dispersion curves are obtained and compared with the results of other theories and with some experimental data.

Przedstawiono mikrostrukturalną teorię drugiego rzędu dla kompozytów sprężystych wzmocnionych włóknami, w celu zbadania własności rozprzestrzeniania się fal harmonicznych w kierunku włókien. Otrzymano krzywe dyspersji, które zostały porównane z wynikami innych teorii i dostępnymi danymi doświadczalnymi.

Представлена микроструктуральная теория второго порядка описывающая композиты армированные волокнами. Теория используется для исследования распространения плоских гармонических волн в направлении волокон. Полученные дисперсные кривые сравниваются с результатами других теорий и с некоторыми экспериментальными данными.

1. Introduction

IN THE CONVENTIONAL method of describing composite media the composite is replaced with a homogeneous anisotropic classical continuum. The geometric arrangement of the phases in a composite material will generally manifest itself as a certain type of anisotropy of this homogeneous continuum whose effective moduli should be determined in terms of the elastic moduli of the constituents and the parameters describing the geometrical layout of the composite. This effective modulus theory cannot account for the dispersion of harmonic waves. Only the lowest (acoustical) modes, and even those without dispersion, can be described by means of the effective modulus theory.

A conceptionally different approach, called the effective stiffness method, was proposed in [1] for the case of a laminated material. A heterogeneous material was transformed into a homogeneous higher-order continuum with microstructure. The method is based on expansions of the displacements in each layer. The coefficients of the expansions constitute the microstructure variables of the theory. Expressing the strain and kinetic energies of the layers in terms of displacement expansions, "smoothing" the resulting expressions to obtain continuous variables, and applying Hamilton's principle results in a continuum theory. A second-order approximation of the displacements in the layers [2, 3] afforded better results for the dispersion of shorter harmonic waves than the first-order approximation used in [1]. For uni-directional fibre-reinforced composites the method was used in [4, 5]; bi-directional fibres were considered in [7]. This approach was also adopted in [6] to find the approximate effective moduli of two-phase elastic composites consisting of a matrix and ellipsoid-, needle- and disc-shaped inclusions.

In [4] uni-directional fibres arranged hexagonally were considered and the displacements in the fibre and in the matrix jacket of a composite element were linear. The aim of the present paper is to obtain a better approximation of dispersion curves for waves propagating in the direction of the fibres using quadratic approximation of the displacements in the matrix.

In Sect. 2 the composite geometry and the displacement expansions are described. In Sect. 3 the strain and kinetic energy densities are defined from which the displacement equations of motion can be obtained through Hamilton's principle. In Sect. 4 the dispersive behaviour of harmonic waves propagating in the direction parallel to the fibres is examined. These results are compared with the results obtained from other theories and with some experimental data.

2. Geometry and kinematics

Let us consider a material consisting of two components: matrix and fibres. Both the matrix and the fibres are linear, elastic, homogeneous and isotropic materials. The Lamé constants and the mass density of the fibres and of the matrix are denoted by λ_1, μ_1, ϱ_1 and λ_2, μ_2, ϱ_2, respectively. The infinitely long fibres are of circular cross-section with a radius r_1 and are arranged in a hexagonal array throughout the matrix material (Fig. 1a).

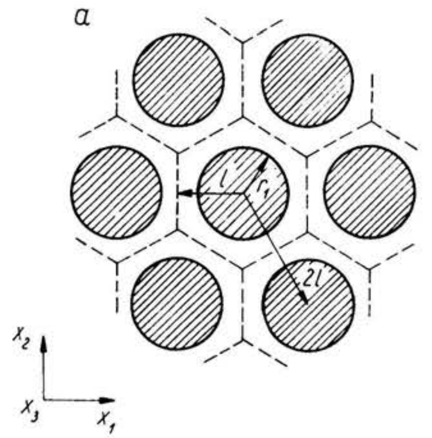

FIG. 1a. Fibre-reinforced composite.

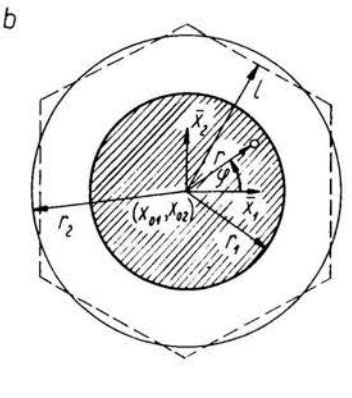

FIG. 1b. Composite element.

The fibres are parallel to the x_3 axis and the distance between them is $2l$. The hexagonal prisms in Fig. 1a are replaced with circular cylinders of the same volume (Fig. 1b). The radius of the cylinder is

$$r_2 = l\sqrt{\frac{2\sqrt{3}}{\pi}} \approx 1.1l.$$

The composite cylinder will be referred to as the composite element.

Let x_1, x_2, x_3 be the global Cartesian coordinates. Let us introduce $\bar{x}_1, \bar{x}_2, \bar{x}_3$ as the local Cartesian coordinates in the composite element with

$$x_1 = x_{01}+\bar{x}_1, \quad x_2 = x_{02}+\bar{x}_2, \quad x_3 = \bar{x}_3,$$

where x_{01}, x_{02} are the global coordinates of the axis of the composite element. Let r, φ denote the local polar coordinates (Fig. 1b), i.e.

$$\bar{x}_1 = r\cos\varphi, \quad \bar{x}_2 = r\sin\varphi.$$

We consider the following displacement distributions in the composite element. In the fibre let the displacement vector $u_i^{(1)}$ have the form

$$u_i^{(1)} = u_{0i}^{(1)}|_1+\bar{x}_1\psi_{1i}|_1+\bar{x}_2\psi_{2i}|_1+\bar{x}_1^2\varphi_{11i}|_1+\bar{x}_2^2\varphi_{22i}|_1+\bar{x}_1\bar{x}_2\varphi_{12i}|_1. \tag{2.1}$$

$u_{0i}^{(1)}$, $\psi_{\alpha i}$, $\varphi_{\alpha\beta i}$, $(\alpha, \beta = 1, 2;\ i = 1, 2, 3)$ depend on x_{01}, x_{02}, x_3 and on time t. $|_1$ means that the value is taken in Point 1 (Fig. 2). If r, φ are used instead of $\bar{x}_\alpha$, it is possible to write Eq. (2.1) in the form

$$u_i^{(1)} = u_{0i}^{(1)}|_1+rU_i^{(1)}|_1+r^2V_i^{(1)}|_1, \tag{2.2}$$

where

$$U_i^{(1)}|_1 = C\psi_{1i}|_1+S\psi_{2i}|_1,$$
$$V_i^{(1)}|_1 = C^2\varphi_{11i}|_1+S^2\varphi_{22i}|_1+SC\varphi_{12i}|_1,$$
$$C = \cos\varphi, \quad S = \sin\varphi.$$

Analogously with Eq. (2.2) we shall assume the vector of displacement in the matrix jacket $u_i^{(2)}$ in the form

$$u_i^{(2)} = u_{0i}^{(2)}|_2+(r-r_2)U_i^{(2)}|_1+(r-r_2)^2V_i^{(2)}|_1. \tag{2.3}$$

$u_{0i}^{(2)}|_2$ refers to the displacement on the circumference of a composite element in the point of local coordinates r_2, φ, i.e. in Point 2 (Fig. 2) and depends on $x_{01}, x_{02}, x_3, r_2, \varphi$ and t.

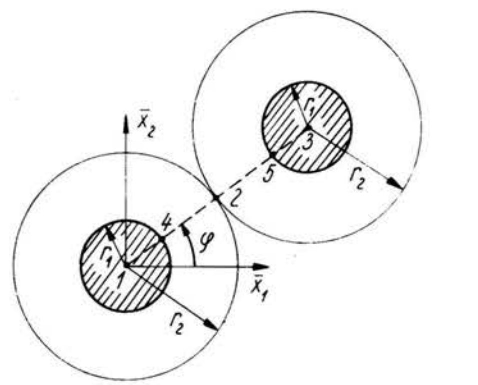

FIG. 2. Two adjoining composite elements.

$U_i^{(2)}$, $V_i^{(2)}$ depend on x_{01}, x_{02}, x_3, φ and t, not depending, however, on r, and will be determined from the continuity of the vector of displacement at the interfaces.

Let us consider two adjoining composite elements (Fig. 2). The elements with the centres in Points 1 and 3, respectively touch in Point 2 of the local coordinates r_2, φ (referred to Point 1). First, we shall express the condition of continuity of displacement in Point 4 of local coordinates r_1, φ. Using Eqs. (2.2) and (2.3), we obtain

$$(2.4)\qquad (r_1-r_2)U_i^{(2)}|_1+(r_1-r_2)^2V_i^{(2)}|_1 = u_{0i}^{(1)}|_1-u_{0i}^{(2)}|_2+r_1U_i^{(1)}|_1+r_1^2V_i^{(1)}|_1.$$

We shall further write the continuity condition of displacement in Point 5 (Fig. 2)

$$u_i^{(1)}|_5 = u_i^{(2)}|_5.$$

From Eq. (2.2) written for the fibre with the axis in Point 3 we obtain in Point 5

$$u_i^{(1)}|_5 = u_{0i}^{(1)}|_3-r_1U_i^{(1)}|_3+r_1^2V_i^{(1)}|_3,$$

where

$$U_i^{(1)}|_3 = C\psi_{1i}|_3+S\psi_{2i}|_3,$$
$$V_i^{(1)}|_3 = C^2\varphi_{11i}|_3+S^2\varphi_{22i}|_3+SC\varphi_{12i}|_3.$$

For $r = 2r_2-r_1$ Eq. (2.3) yields

$$u_i^{(2)}|_5 = u_{0i}^{(2)}|_2+(r_2-r_1)U_i^{(2)}|_1+(r_2-r_1)^2V_i^{(2)}|_1.$$

The continuity condition of displacement in Point 5 is, therefore,

$$(2.5)\qquad (r_2-r_1)U_i^{(2)}|_1+(r_2-r_1)^2V_i^{(2)}|_1 = u_{0i}^{(1)}|_3-u_{0i}^{(2)}|_2-r_1U_i^{(1)}|_3+r_1^2V_i^{(1)}|_3.$$

Let us note that for simplification purposes we have assumed in deducing Eq. (2.5) that the distance of the axes of adjoining composite elements is $2r_2$ instead of $2l$ (see Fig. 2). Equation (2.4) is valid identically for all φ. The composite element with a centre in Point 1 adjoins six composite elements (see Fig. 1a). Consequently, the continuity condition (2.5) is valid for six angles, viz. φ_0, $\varphi_0+\frac{\pi}{3}$, $\varphi_0+\frac{2}{3}\pi$, $\varphi_0+\pi$, $\varphi_0+\frac{4}{3}\pi$, $\varphi_0+\frac{5}{3}\pi$. From the macroscopic point of view, however, no angle φ_0 is preferred and we shall require that Eq. (2.5) be valid in the same way as Eq. (2.4) for all φ. Let us note that $u_i^{(2)}$ changes smoothly in the transition to the adjoining composite element (i.e. over Point 2 in Fig. 2). The sum and the difference of Eqs. (2.4) and (2.5) yields $U_i^{(2)}|_1$, $V_i^{(2)}|_1$ in the form

$$(2.6)\qquad \begin{aligned} 2(r_1-r_2)U_i^{(2)}|_1 &= u_{0i}^{(1)}|_1-u_{0i}^{(1)}|_3+r_1(U_i^{(1)}|_1+U_i^{(1)}|_3)+r_1^2(V_i^{(1)}|_1-V_i^{(1)}|_3),\\ 2(r_1-r_2)^2V_i^{(2)}|_1 &= u_{0i}^{(1)}|_1+u_{0i}^{(1)}|_3-2u_{0i}^{(2)}|_2+r_1(U_i^{(1)}|_1-U_i^{(1)}|_3)+r_1^2(V_i^{(1)}|_1+V_i^{(1)}|_3).\end{aligned}$$

So far $u_{0i}^{(1)}$, $u_{0i}^{(2)}$, $U_i^{(1)}$, $U_i^{(2)}$, $V_i^{(1)}$ and $V_i^{(2)}$ have been defined in the axes or on the surfaces of composite elements only. Our aim is to describe the microscopically heterogeneous material continuously. We assume that in every macroscopic point both phases and material boundaries exist simultaneously and it is necessary that Eqs. (2.1)–(2.6) be valid in every point. Therefore, we shall replace $u_{0i}^{(1)}$, $u_{0i}^{(2)}$, $U_i^{(1)}$, $U_i^{(2)}$, $V_i^{(1)}$, $V_i^{(2)}$ with continuous functions depending on the continuous variables x_1, x_2 instead of the discrete variables x_{01}, x_{02}. Now we can express all functions in Points 2 and 3 by means of Taylor's expansion by the values of these functions and their derivatives in Point 1. After that, Eq. (2.6) will acquire the form

$$(2.7)\qquad \begin{aligned} U_i^{(2)} &= CG_{1i}+SG_{2i}+C^2r_2G_{1i,1}+S^2r_2G_{2i,2}+SCr_2(G_{1i,2}+G_{2i,1}),\\ V_i^{(2)} &= H_i+Cr_2H_{i,1}+Sr_2H_{i,2}\\ &\quad +C^2\left[\frac{1}{\eta'}G_{1i,1}+K_{11i}\right]+S^2\left[\frac{1}{\eta'}G_{2i,2}+K_{22i}\right]+SC\left[\frac{1}{\eta'}(G_{1i,2}+G_{2i,1})+2K_{12i}\right],\end{aligned}$$

where

$$G_{\alpha i} = \frac{1}{\eta'}(u^{(1)}_{0i,\alpha} - \eta\psi_{\alpha i}), \qquad H_i = \frac{1}{(r_2 - r_1)^2}(u^{(1)}_{0i} - u^{(2)}_{0i}),$$

$$K_{\alpha\beta i} = \frac{1}{\eta'^2}\left(\eta^2\varphi_{\alpha\beta i} - \frac{1}{2}u^{(2)}_{0i,\alpha\beta}\right), \qquad \eta = \frac{r_1}{r_2}, \qquad \eta' = 1-\eta,$$

$$\alpha, \beta = 1, 2, \qquad i = 1, 2, 3.$$

All functions are taken in Point 1. The comma with the following index denotes a partial derivative with respect to the corresponding global Cartesian coordinate. To express $U^{(2)}_i$, $V^{(2)}_i$ in Eq. (2.7) as many terms in Taylor's expansion were taken as were required to obtain, by substituting Eq. (2.7) into Eq. (2.3), $u^{(2)}_i$ with the accuracy of small quantities of the second order (in r_2). Thus Eqs. (2.2), (2.3) and (2.7) represent an approximation of the second order.

Apart from the conditions of the continuity of displacement also the conditions of the continuity of the stress vector should be satisfied at the interfaces. We could supplement Eq. (2.3) with the term

$$(r - r_2)^3 W^{(2)}_i|_1$$

and determine, from the conditions of the displacement continuity and stress vector continuity in Points 4 and 5. twelve functions $u^{(2)}_{0i}$, $U^{(2)}_i$, $V^{(2)}_i$, $W^{(2)}_i$ in dependence on $u^{(1)}_{0i}$, $\psi_{\alpha i}$, $\varphi_{\alpha\beta i}$. We should obtain, however, very complicated expressions. It was found in [2] that in the second-order theory of laminated materials the effect of the stress boundary conditions at the interfaces on the dispersion curves was negligible. For this reason, and for the sake of simplicity, we shall neglect the stress boundary conditions at the interfaces in our case as well.

In technical practice, as a rule, all fibres have a rigidity of a higher order than the matrix. For this case another simplification of the second-order approximation can be made. In [3] a second-order approximation of the displacement in the layers of a laminated material was considered. Figure 4 in [3] shows that for longitudinal waves propagating along the layers the distribution of the displacement across the thickness of the layers with the rigidity of a higher order appears in this approximation approximately linear (although, according to Fig. 5 in [3], the accurate solution for the third mode is by far not linear). It seems also that for a fibrous material with very rigid fibres the assumption

$$V^{(1)}_i = 0 \tag{2.8}$$

will not result in any major errors. If we accept the condition (2.8), the number of independent kinematic quantities will be reduced by 9, as Eq. (2.8) means that

$$\varphi_{\alpha\beta i} = 0, \qquad \alpha, \beta = 1, 2; \qquad i = 1, 2, 3.$$

Let us note that if we applied, apart from Eq. (2.8), also

$$V^{(2)}_i = 0, \tag{2.9}$$

we would obtain the approximations (2.1) and (2.2) given in [4]. If Eqs. (2.8) and (2.9) hold, we see from Eq. $(2.6)_2$ that with the accuracy of the infinitely small quantities of the first order it follows from the conditions of the continuity of displacement in Point 4 and 5 that

$$u_{0i}^{(1)} = u_{0i}^{(2)}. \tag{2.10}$$

Thus the a priori assumption (2.10) made for the linear approximations (2.1) and (2.2) in [4] is substantiated. However, for the second-order approximation Eq. (2.10) cannot be generally accepted.

Further on we shall confine our calculations of dispersion curves on the wave propagating in the direction of x_3, i.e. along the fibres. If Eq. (2.8) holds, then Eqs. (2.2), (2.3) and (2.7) for the motion in the direction of x_3 will be simplified to the form

$$\begin{aligned} u_i^{(1)} &= u_{0i}^{(1)} + rU_i^{(1)}, \\ u_i^{(2)} &= u_{0i}^{(2)} + (r-r_2)U_i^{(2)} + (r-r_2)^2 V_i^{(2)}, \end{aligned} \tag{2.11}$$

where

$$U_i^{(1)} = C\psi_{1i} + S\psi_{2i}, \qquad U_i^{(2)} = CG_{1i} + SG_{2i},$$

$$V_i^{(2)} = \frac{1}{(r_2-r_1)^2}(u_{0i}^{(1)} - u_{0i}^{(2)}), \qquad G_{\alpha i} = -\frac{\eta}{\eta'}\psi_{\alpha i},$$

$$C = \cos\varphi, \qquad S = \sin\varphi, \qquad \eta = \frac{r_1}{r_2}, \qquad \eta' = 1-\eta, \qquad \alpha = 1, 2; \qquad i = 1, 2, 3.$$

In Eqs. (2.11) there are twelve independently variable kinematic quantities $u_{0i}^{(1)}$, $u_{0i}^{(2)}$, ψ_{1i}, ψ_{2i}; $i = 1, 2, 3$. For the motion in the direction of x_3 the second-order approximation (2.11) differs from the linear approximation in [4] by that

$$V_i^{(2)} \neq 0$$

and $u_{0i}^{(1)}$, $u_{0i}^{(2)}$ are independent.

3. Equations of motion

The strain energy per unit volume of the composite medium is defined by

$$W = W^{(1)} + W^{(2)},$$

$$W^{(\alpha)} = \frac{1}{\pi r_2^2}\iint_{F^{(\alpha)}}\left[\frac{1}{2}\lambda_\alpha \varepsilon_{ii}^{(\alpha)}\varepsilon_{kk}^{(\alpha)} + \mu_\alpha \varepsilon_{ij}^{(\alpha)}\varepsilon_{ij}^{(\alpha)}\right]d\bar{x}_1\, d\bar{x}_2, \tag{3.1}$$

$$\varepsilon_{ij}^{(\alpha)} = \frac{1}{2}(u_{i,j}^{(\alpha)} + u_{j,i}^{(\alpha)}), \qquad \alpha = 1, 2.$$

Here differentiation is taken with regard to the local coordinates $\bar{x}_1$, $\bar{x}_2$, $\bar{x}_3$. Summation of pairs of identical Latin indices over 1, 2, 3 is implied. $F^{(1)}$ in Eqs. (3.1) is the part of the cross-section of the composite element belonging to the fibre, $F^{(2)}$ is that belonging to the matrix.

The kinetic energy per unit volume of the composite material is defined by

(3.2)
$$K = K^{(1)} + K^{(2)},$$
$$K^{(\alpha)} = \frac{1}{\pi r_2^2} \iint\limits_{F^{(\alpha)}} \frac{1}{2} \varrho_\alpha \dot{u}_i^{(\alpha)} \dot{u}_i^{(\alpha)} d\bar{x}_1 d\bar{x}_2; \qquad \alpha = 1, 2.$$

The dot above a quantity denotes the derivative with regard to time.

Using Eqs. (2.11) we obtain $W^{(1)}$, $W^{(2)}$ from Eqs. (3.1) in the form

$$W^{(1)} = \eta^2 \Big\{ \frac{\lambda_1}{2} \Big[(\psi_{11} + \psi_{22})^2 + u_{03,3}^{(1)2} + 2u_{03,3}^{(1)} (\psi_{11} + \psi_{22}) + \frac{r_1^2}{4} (\psi_{13,3}^2 + \psi_{23,3}^2) \Big]$$
$$+ \mu_1 \Big[\psi_{11}^2 + \psi_{22}^2 + u_{03,3}^{(1)2} + \frac{1}{2} (\psi_{12} + \psi_{21})^2 + \frac{1}{2} (\psi_{13} + u_{01,3}^{(1)})^2 + \frac{1}{2} (\psi_{23} + u_{02,3}^{(1)})^2$$
$$+ \frac{r_1^2}{4} \Big(\psi_{13,3}^2 + \psi_{23,3}^2 + \frac{1}{2} \psi_{11,3}^2 + \frac{1}{2} \psi_{12,3}^2 + \frac{1}{2} \psi_{21,3}^2 + \frac{1}{2} \psi_{22,3}^2 \Big) \Big] \Big\},$$

(3.3)
$$W^{(2)} = \frac{\lambda_2}{2} \{ (3V - \eta^2)(\psi_{11}^2 + \psi_{22}^2) + (2V - \eta^2) \psi_{11} \psi_{22} + V(\psi_{12} + \psi_{21})^2$$
$$- 2\eta^2 u_{03,3}^{(2)} (\psi_{11} + \psi_{22}) + Y(\psi_{13,3}^2 + \psi_{23,3}^2) + (1 - \eta^2) u_{03,3}^{(2)2} + N(\psi_{13,3} U_1 + \psi_{23,3} U_2)$$
$$+ Z(U_1^2 + U_2^2) \} + \frac{\mu_2}{2} \{ (3V - \eta^2)(2\psi_{11}^2 + 2\psi_{22}^2 + \psi_{12}^2 + \psi_{21}^2 + \psi_{13}^2 + \psi_{23}^2)$$
$$+ V(2\psi_{12}^2 + 2\psi_{21}^2 + \psi_{11}^2 + \psi_{22}^2 + \psi_{13}^2 + \psi_{23}^2 + 2\psi_{11}\psi_{22} + 2\psi_{12}\psi_{21}) - 2\eta^2 (\psi_{12}\psi_{21}$$
$$+ u_{01,3}^{(2)} \psi_{13} + u_{02,3}^{(2)} \psi_{23}) + (1 - \eta^2)(2u_{03,3}^{(2)2} + u_{01,3}^{(2)2} + u_{02,3}^{(2)2}) + Y(2\psi_{13,3}^2 + 2\psi_{23,3}^2$$
$$+ \psi_{11,3}^2 + \psi_{12,3}^2 + \psi_{21,3}^2 + \psi_{22,3}^2) + P(2u_{03,3}^{(2)} U_{3,3} + u_{01,3}^{(2)} U_{1,3} + u_{02,3}^{(2)} U_{2,3})$$
$$+ Q(2U_{3,3}^2 + U_{1,3}^2 + U_{2,3}^2) + N(\psi_{11,3} U_3 + \psi_{22,3} U_3)$$
$$+ R(\psi_{13} U_{1,3} + \psi_{23} U_{2,3}) + Z(3U_1^2 + 3U_2^2 + 3U_3^2) \}$$

and K from Eqs. (3.2) in the form

(3.4)
$$K = \frac{1}{2} \{ \varrho_1 \eta^2 \dot{u}_{0i}^{(1)} \dot{u}_{0i}^{(1)} + \varrho_2 (1 - \eta^2) \dot{u}_{0i}^{(2)} \dot{u}_{0i}^{(2)} + I(\dot{\psi}_{1i} \dot{\psi}_{1i} + \dot{\psi}_{2i} \dot{\psi}_{2i}) + \varrho_2 P \dot{u}_{0i}^{(2)} \dot{U}_i + \varrho_2 Q \dot{U}_i \dot{U}_i \}.$$

We have introduced the notation

$$N = -\frac{1}{3} \eta (1 + 3\eta), \qquad P = \frac{1}{3} \eta' (1 + 3\eta),$$
$$Q = \frac{1}{15} \eta' (1 + 5\eta), \qquad R = \frac{1}{3} \eta (1 - 3\eta),$$
$$U_i = u_{0i}^{(1)} - u_{0i}^{(2)}, \qquad V = \frac{1}{4} \frac{\eta^2}{\eta'^2} \lg \frac{1}{\eta},$$
$$Y = \frac{r_1^2}{12} \eta' (1 + 3\eta), \qquad Z = \frac{1 + 3\eta}{3\eta'} \frac{\eta^2}{r_1^2},$$
$$I = \frac{r_1^2}{4} \Big[\varrho_1 \eta^2 + \frac{\varrho_2}{3} \eta' (1 + 3\eta) \Big].$$

Let V denote a fixed regular region and t_1, t_2 fixed times. For independent variations of $\delta u_{0i}^{(\alpha)}$, $\delta \psi_{\alpha i} (\alpha = 1, 2;\ i = 1, 2, 3)$ for which

$$\delta u_{0i}^{(\alpha)} = \delta \psi_{\alpha i} = 0$$

on the surface S of the region V, Hamilton's principle is of the form

$$\delta \int_{t_1}^{t_2} \int_V (K - W)\, dV\, dt = 0. \tag{3.5}$$

The sought equations of motion of the composite material are the Euler equations for the variational principle (3.5) which have, in this particular case, the form

$$\frac{\partial W}{\partial f} - \frac{\partial}{\partial x_3}\left(\frac{\partial W}{\partial f_{,3}}\right) + \frac{\partial}{\partial t}\left(\frac{\partial K}{\partial \dot{f}}\right) = 0. \tag{3.6}$$

In Eq. (3.6) f represents the twelve functions $u_{0i}^{(\alpha)}$, $\psi_{\alpha i}$ $(\alpha = 1, 2,\ \ i = 1, 2, 3)$.

4. Wave propagation results

The principal aim of this paper is to find the approximate dispersion curves for plane harmonic wave propagation along the fibres, i.e. in the direction of x_3.

Let us assume the solution of the equations of motion (3.6) in the form

$$u_{0j}^{(\alpha)} = U_{0j}^{(\alpha)} e^{ik(x_3 - ct)}, \quad \psi_{\alpha j} = \Psi_{\alpha j} e^{ik(x_3 - ct)}, \qquad \alpha = 1, 2, \quad j = 1, 2, 3. \tag{4.1}$$

$U_{0j}^{(\alpha)}$, $\Psi_{\alpha j}$ are constant amplitudes, k is the wave number, c is the phase velocity and i stands for the imaginary unit. After substituting the relations (4.1) into the equations of motion (3.6), we obtain a homogeneous system of linear equations for $U_{0j}^{(\alpha)}$, $\Psi_{\alpha j}$. The system of twelve equations decomposes into four systems corresponding with the individual waves. The first system contains $U_{01}^{(1)}$, $U_{01}^{(2)}$, Ψ_{13}, the second $U_{02}^{(1)}$ $U_{02}^{(2)}$, Ψ_{23}, the third $U_{03}^{(1)}$, $U_{03}^{(2)}$, Ψ_{11}, Ψ_{22} and the fourth Ψ_{12}, Ψ_{21}. The first two systems of equations describe the transverse waves, the third represents the longitudinal wave and the fourth the twisting microwave.

For the transverse waves the dispersion relations turn out to be

$$\begin{vmatrix} c_1 - c^2 c_5 + k^{-2} c_3, & c_2 - c^2 c_6 - k^{-2} c_3, & c_4 \\ c_2 - c^2 c_6 - k^{-2} c_3, & c_7 - c^2 c_9 + k^{-2} c_3, & c_8 \\ c_4, & c_8, & c_{11} + k^2 c_{10} - k^2 c^2 I \end{vmatrix} = 0, \tag{4.2}$$

where

$$c_1 = \left[\gamma \eta^2 + \frac{\eta'}{15}(1 + 5\eta)\right]\mu_2, \qquad c_2 = \frac{\eta'}{30}(3 + 5\eta)\mu_2,$$

$$c_3 = \frac{\eta^2(1 + 3\eta)}{3\eta'}(\delta_2 + 1)\frac{\mu_2}{r_1^2},$$

$$c_4 = \left\{\gamma \eta^2 + \frac{\eta}{6}[(\delta_2 - 2)(1 + 3\eta) + (1 - 3\eta)]\right\}\mu_2,$$

$$c_5 = \left[\vartheta\eta^2 + \frac{\eta'}{15}(1+5\eta)\right]\varrho_2, \quad c_6 = \frac{\eta'}{30}(3+5\eta)\varrho_2,$$

$$c_7 = \left[(1-\eta^2) - \frac{2}{15}\eta'(2+5\eta)\right]\mu_2,$$

$$c_8 = \left\{-\eta^2 - \frac{\eta}{6}[(\delta_2-2)(1+3\eta)]\right\}\mu_2,$$

$$c_9 = \left[(1-\eta^2) - \frac{2}{15}\eta'(2+5\eta)\right]\varrho_2,$$

$$c_{10} = \frac{r_1^2}{4}\left[\eta^2\delta_1\gamma + \frac{\delta_2\eta'}{3}(1+3\eta)\right]\mu_2,$$

$$c_{11} = [\eta^2(\gamma-1)+4V]\mu_2, \quad J = \frac{r_1^2}{4}\left[\vartheta\eta^2 + \frac{\eta'}{3}(1+3\eta)\right]\varrho_2,$$

$$\vartheta = \frac{\varrho_1}{\varrho_2}, \quad \gamma = \frac{\mu_1}{\mu_2}, \quad \delta_\alpha = \frac{2(1-\nu_\alpha)}{1-2\nu_\alpha}, \quad \nu_\alpha = \frac{\lambda_\alpha}{2(\lambda_\alpha+\mu_\alpha)}, \quad \alpha = 1, 2.$$

The phase velocity c_0 of very long waves is obtained from Eqs. (4.2) for $k \to 0$

$$c_0^2 = \frac{c_{11}(c_1+2c_2+c_7)-(c_4+c_8)^2}{c_{11}(c_5+2c_6+c_9)}. \tag{4.3}$$

For the longitudinal waves due to axial symmetry with regard to x_3 the dispersion relation is

$$\begin{vmatrix} d_1 - c^2d_5 + k^{-2}d_3, & d_2 - c^2d_6 - k^{-2}d_3, & 2d_4 \\ d_2 - c^2d_6 - k^{-2}d_3, & d_7 - c^2d_9 + k^{-2}d_3, & 2d_3 \\ d_4, & d_8, & d_{11} + k^2d_{10} - k^2c^2I \end{vmatrix} = 0, \tag{4.4}$$

where

$$d_1 = \left[\delta_1\eta^2\gamma + \frac{2}{15}\eta'(1+5\eta)\right]\mu_2, \quad d_2 = \frac{\eta'}{15}(3+5\eta)\mu_2,$$

$$d_3 = \frac{2}{3}\frac{\eta^2}{\eta'}(1+3\eta)\frac{\mu_2}{r_1^2}, \quad d_4 = \left[(\delta_1-2)\eta^2\gamma + \frac{\eta}{6}(1+3\eta)\right]\mu_2,$$

$$d_5 = \left[\eta^2\vartheta + \frac{\eta'}{15}(1+5\eta)\right]\varrho_2, \quad d_6 = \frac{\eta'}{30}(3+5\eta)\varrho_2,$$

$$d_7 = \left[(1-\eta^2)\delta_2 - \frac{4}{15}\eta'(2+5\eta)\right]\mu_2,$$

$$d_8 = -\left[\eta^2(\delta_2-2) + \frac{\eta}{6}(1+3\eta)\right]\mu_2,$$

$$d_9 = \left[(1-\eta^2) - \frac{2}{15}\eta'(2+5\eta)\right]\varrho_2, \quad d_{10} = \frac{r_1^2}{4}\left[\eta^2\gamma + \frac{\eta'}{3}(1+3\eta)\right]\mu_2,$$

$$d_{11} = 2[\eta^2(\delta_1\gamma-\delta_2) - \eta^2(\gamma-1) + 2V\delta_2]\mu_2.$$

c_0 is of the form

$$c_0^2 = \frac{d_{11}(d_1+2d_2+d_7)-2(d_4+d_8)^2}{d_{11}(d_5+2d_6+d_9)}. \tag{4.5}$$

We shall easily ascertain that these c_0 merge with phase velocities of the corresponding waves of long wave lengths obtained in the linear approximation in [4] (compare Eqs. (4.3) and (4.5) with Eqs. (4.3) and (4.4) in [4]), respectively.

The dispersion relations (4.2) and (4.4) will be calculated numerically for the volume fraction of fibres $\eta^2 = 0.6$ and for the material parameters

$$\gamma = \frac{\mu_1}{\mu_2} = 100, \quad \vartheta = \frac{\varrho_1}{\varrho_2} = 3, \quad \nu_1 = \nu_2 = 0.3.$$

Instead of c, k and the circular frequency $\omega = ck$, the dimensionless quantities will be used, viz.

$$\beta = c\left(\frac{\varrho_2}{\mu_2}\right)^{1/2}, \quad \xi = kr_1, \quad \Omega = \beta\xi.$$

Figures 3 and 4 show the calculated dispersion curves of transverse waves, Figs. 5 and 6 being concerned with longitudinal waves. Solid lines show the dispersion curves of the second-order approximations (4.2) and (4.4), dashed lines those of the first-order approximation [4]. Dotted lines in Figs. 4 and 5 are the results of the calculations from Eqs. (3.16) and (3.54) from [5]. The dash-and-dot curves in Figs. 4 and 5 are taken over from [8]. In [5] and [8] the parallel fibres were arranged in a quadratic array.

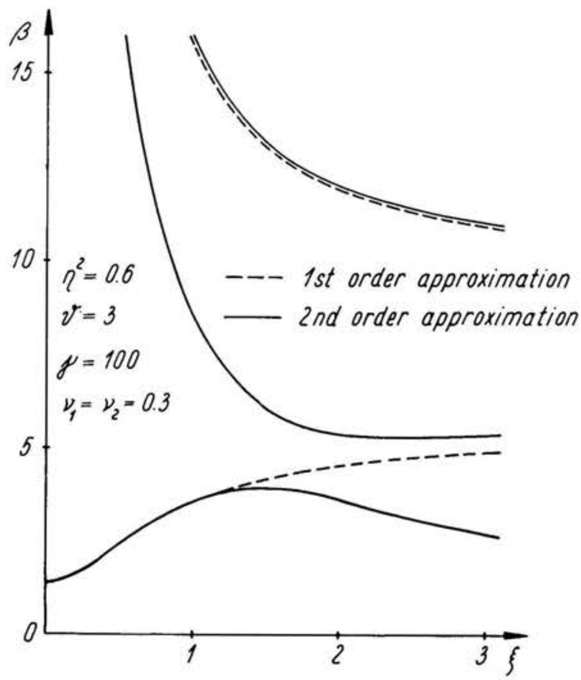

FIG. 3. Dispersion curves for transverse waves.

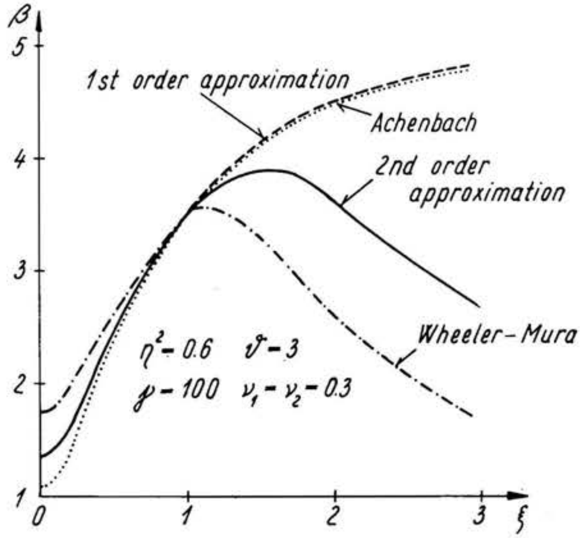

FIG. 4. Lowest (acoustical) transverse mode.

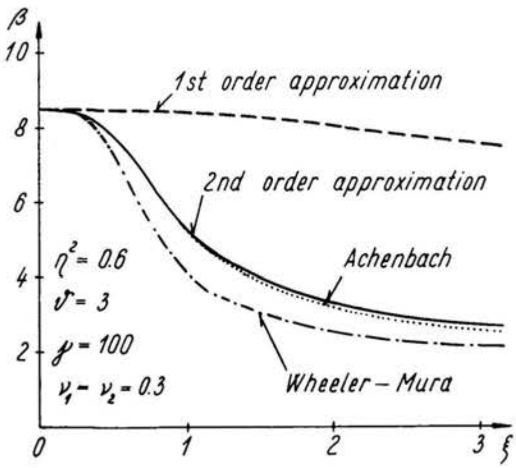

FIG. 5. Lowest (acoustical) longitudinal mode.

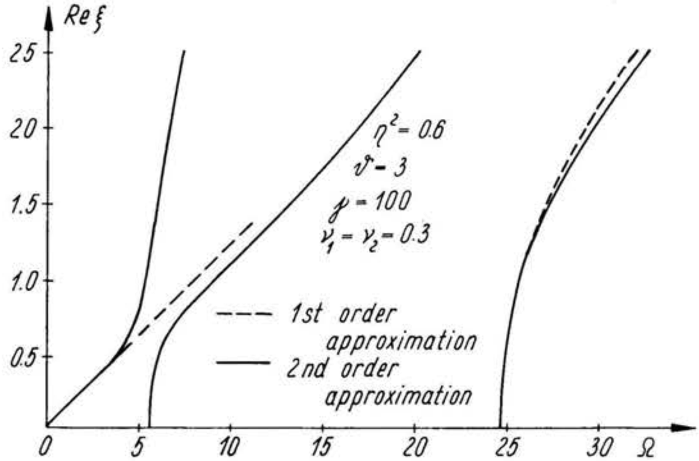

FIG. 6. Dispersion curves for longitudinal waves.

In Fig. 3 the results of both approximations are compared for transverse waves. The dimensionless phase velocities β are plotted against the dimensionless wave numbers ξ. The second-order approximation gives three modes of propagation, the first-order approximation gives only two of them. The first (acoustical) modes, i.e. the lowest dispersion curves in Fig. 3, are very near in the first and the second order approximations for small ξ (long waves); for $\xi \to 0$ the dimensionless phase velocities β merge. With the increasing ξ (i.e. with the shortening of the wave length) the differences increase. The remaining (optical) mode of the first-order approximation for all ξ is very near the highest optical mode of the second-order approximation. The intermediate mode of the second-order approximation has no partner in the first approximation. Figure 4 shows in greater detail the acoustical modes of the transverse waves. The dash-and-dot curve calculated in [8] by the Ritz method has a qualitatively identical course with the second-order approximation curve; however, for small ξ it affords higher values. The dotted curve calculated from Eq. (3.16) in [5] has a similar course as the first-order approximation curve; however, for small ξ it affords lower values. β calculated for $\xi \to 0$ from both of our approximations corresponds, with a high accuracy, to the value calculated by means of the effective static modulus found in [10] and the effective dynamic modulus obtained in [11] for the quadratic array of fibres. For long transverse waves, β calculated according to [5] appears too low and β obtained in [8] too high. Figure 5 shows the first (acoustic) modes of the longitudinal waves. The second-order approximation curve is very near the curve calculated from Eq. (3.54) in [5] and that taken from [8] and represents a great improvement in comparison with the first-order approximation curve. For $\xi \to 0$ all curves merge. In Fig. 6 the real part of ξ is plotted against the dimensionless frequency Ω for the longitudinal waves. In the second-order approximation the first (acoustic) mode is real for all Ω. The second and the third modes are imaginary (standing waves), until cut-off frequencies have been attained after which they become real. The two modes of the first-order approximation (dashed lines) are near the second-order approximation for long waves (small ξ).

In [9] results of experiments are given with a boron-epoxy composite. The group velocity c_g defined by

$$c_g = \frac{d\omega}{dk} = c + k\frac{dc}{dk} \tag{4.6}$$

was measured for a number of frequencies $f = \omega/2\pi$. For the case studied in [9] it is

$$\eta^2 = 0.54, \quad \gamma = 88.1, \quad \vartheta = 2.13, \quad \nu_1 = 0.2, \quad \nu_2 = 0.4.$$

For these parameters in Figs. 7 and 8 the group velocity c_g [in/μ s] calculated from Eqs. (4.6), (4.2) and (4.4) is plotted in solid lines against the frequency f [MHz]. The dashed line in Fig. 7 is taken from [5], Fig. 3. It can be seen that for transverse as well as for longitudinal waves the second-order approximation results for $f < 5$ MHz are in good accordance with experimental results.

The above comparisons show that the second-order approximation of displacements in the matrix affords considerably better results for the propagation of waves of shorter wave lengths in the direction of the fibres than the linear (first-order) approximation presented in [4].

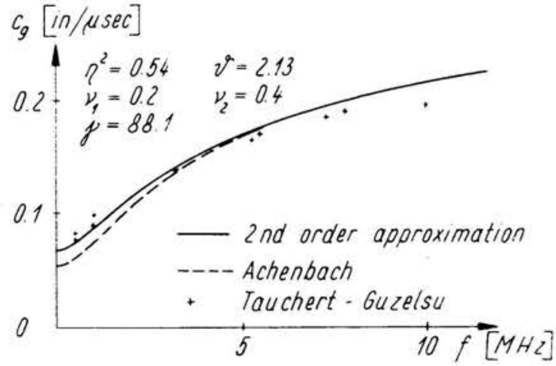

FIG. 7. Group velocity vs. frequency for transverse waves.

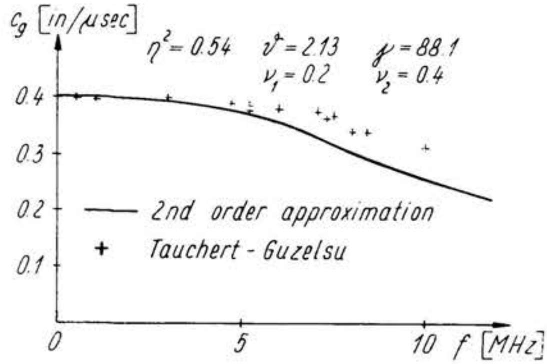

FIG. 8. Group velocity vs. frequency for longitudinal waves.

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