

Coupling of dissipative mechanisms of viscoplastic flow(*)

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THE MAIN objective of the paper is to describe coupling effects between dissipative mechanisms of an elastic-viscoplastic material. The other aim is to construct a reasonable and physically justified theory of viscoplasticity within the framework of a material structure with internal state variables. In particular the internal friction, the thermally activated and the high velocity dislocation damping mechanisms are considered. A discussion of coupling of dissipative effects for these mechanisms is given. Two groups of internal state variables are introduced. It has been shown that the group of internal state variables describing dissipative effects of the internal friction mechanism can be eliminated. In this case, of course, the evolution equations for a plastic flow mechanism are modified in such a way that the influence of internal friction effects are taken into account. The evolution equations for internal state variables describing the main mechanism take the form of integral or integro-differential equations. One dimensional examples are considered. Experimental results for titanium, aluminium and copper are used to choose functions involved in the description. The method is developed to generalize to three-dimensional case the procedure proposed.

Głównym celem pracy jest opis efektów sprzężeń mechanizmów dysypacyjnych materiału sprężysto-lepkoplastycznego. Celem następnym jest opracowanie fizykalnie uzasadnionej teorii lepkoplastyczności w ramach struktury materialnej z parametrami wewnętrznymi. W szczególności rozważono mechanizm tarcia wewnętrznego, mechanizm termicznie aktywowany oraz mechanizm tłumienia ruchu dyslokacji. Przeprowadzono dyskusję sprzężeń efektów dysypacyjnych dla tych mechanizmów. Wprowadzono dwie grupy parametrów wewnętrznych. Wykazano, że grupa parametrów wewnętrznych opisująca efekty dysypacyjne wynikające z mechanizmu tarcia wewnętrznego może być wyeliminowana z rozważań. W tym przypadku, oczywiście, równania ewolucji dla mechanizmu plastycznego płynięcia zostają zmodyfikowane w taki sposób, że wpływ efektów tarcia wewnętrznego jest uwzględniony. Równania ewolucji dla parametrów wewnętrznych opisujących główny mechanizm dysypacyjny przyjmują postać równań całkowych lub całkowo-różniczkowych. Rozważono przykłady jednowymiarowe. Posłużono się rezultatami doświadczalnymi dla tytanu, aluminium i miedzi w celu określenia funkcji materiałowych wprowadzonych do opisu. Pokazano metodę uogólnienia opracowanego formalizmu na przypadki trójwymiarowe.

Главной целью работы является описание эффектов сопряжений диссипативных механизмов упруго-вязкопластического материала. Дальшей целью является разработка физически обоснованной теории вязкопластичности в рамках материальной структуры с внутренними параметрами. В частности рассмотрен механизм внутреннего трения, механизм термической активации, а также механизм затухания движения дислокаций. Проведено обсуждение сопряжений диссипативных эффектов для этих механизмов. Введены две группы внутренних параметров. Показано, что группа внутренних параметров, описывающая диссипативные эффекты, вытекающие из механизма внутреннего трения, может быть исключена из рассуждений. В этом случае, конечно, уравнения эволюции для механизма пластического течения будут модифицированы таким образом, что влияние эффектов внутреннего трения учитывается. Уравнения эволюции для внутренних параметров, описывающих главный диссипативный механизм, принимают вид интегральных или интегро-дифференциальных уравнений. Рассмотрены одномерные примеры. Послуживаются экспериментальными результатами для титана, алюминия и меди с целью определения материальных функций введенных в описание. Показан метод обобщения разработанного формализма на трехмерные случаи.

(*) The paper has been presented at the 14-th IUTAM Congress, Delft, 30 August—4 September 1976.

1. Introduction

RECENT experimental investigations have shown that the viscoplastic flow in single crystals of metals and in polycrystalline metals is governed by several dissipative mechanisms⁽¹⁾. Physical analysis based on the dynamics of dislocations suggested that two mechanisms, namely the thermally activated process and the phonon damping effects are most important for explaining of the strain rate and temperature sensitivity of metals.

It has been observed that coupling phenomena between the main mechanisms of plastic flow and the internal friction mechanism have a great influence on the response of metals to some processes of deformation and temperature changes.

The main objective of this paper is to develop the procedure of description of coupling of the dissipative effects observed.

It has been shown that this formalism can be achieved within the framework of the material structure with internal state variables. Two groups of internal state variables are introduced and thermodynamic requirements are fulfilled. It has been proved that the group of internal state variables describing the dissipative effects of the internal friction mechanism can be eliminated. As a result of this procedure the evolution equations for plastic flow mechanisms are modified in such a way that the influence of internal friction effects on plastic flow are taken into consideration. The evolution equations for the internal state variables describing the main mechanism take the form of integro-differential equations (or integral equations).

Based on one-dimensional physical analysis and available experimental data a set of rules of interpretation for internal state variables introduced is given. Two approximate descriptions of rate sensitive plastic flow are proposed and examples of coupling phenomena are considered. Experimental results for titanium, aluminium and copper are used to choose the material functions involved in the description.

The method is shown to generalize the formalism proposed to three-dimensional case.

2. The material structure with internal state variables

In the previous paper [21] the author developed a general thermodynamic theory of dissipative materials. In this theory an important role plays notion of the intrinsic state⁽²⁾ in a particle X of a body \mathcal{B} . The intrinsic state σ of a particle X is defined as a pair—the local deformation-temperature configuration $P(t) = (C(t), \vartheta(t), \nabla\vartheta(t))$ of a particle X and its method of preparation $A(t)$, i.e.

$$(2.1) \quad \sigma = (P(t), A(t)).$$

A method of preparation $A(t)$ of the deformation-temperature configuration of a particle X represents the way of inserting the additional information required to define uniquely

⁽¹⁾ The analysis of dissipative mechanisms has been given for mild steel by A. R. ROSENFELD and G. T. HAHN [24] and J. D. CAMPBELL and W. G. FERGUSON [1] and for aluminium by C. K. H. DHARAN and F. E. HAUSER [4]. Cf. also the discussion of this subject presented by the author [18, 19 and 20].

⁽²⁾ The precise definition of the intrinsic state for a pure mechanical process was first given by P. PERZYNA and W. KOSIŃSKI [22].

the intrinsic state of a particle X and is needed to describe the internal dissipation of a material.

By particular realization of a method of preparation an internal state variable material structure can be constructed. It is postulated that $A(t)$ is an element of a finite dimensional vector space \mathcal{V}_n .

The principle of determinism for the material structure with internal state variables is expressed by the constitutive equations

$$(2.2) \quad Z(t) = \hat{S}(P(t), A(t)),$$

and can be stated as follows: A unique value of the response $Z(t)$ of a material at X , i.e. unique values of the free energy $\psi(t)$, the entropy $\eta(t)$, the Piola-Kirchhoff stress tensor $T(t)$, and the heat flux vector $q(t)$, corresponds to every intrinsic state σ .

The mapping \hat{S} is called the response function and represents the free energy response function $\hat{\Psi}$, the entropy response function \hat{N} , the stress response function \hat{T} and the heat response function \hat{Q} , i.e.

$$(2.3) \quad \hat{S} \equiv \{\hat{\Psi}, \hat{N}, \hat{T}, \hat{Q}\}.$$

The evolution for this material structure is given by the initial-value problem for an internal state variable $A(t)$:

$$(2.4) \quad \frac{d}{dt} A(t) = \hat{A}(P(t), A(t)), \quad A(0) = A_0.$$

It is assumed that the initial-value problem (2.4) has a unique solution in the form

$$(2.5) \quad A(t) = \mathfrak{F}(P_{[0,t]}, A_0).$$

The dissipation principle requires that the results

$$(2.6) \quad \begin{aligned} \partial_{\nabla\theta(t)} \hat{\Psi}(\cdot) &= 0, \quad T(t) = 2\rho_0 \partial_{C(t)} \hat{\Psi}(\cdot), \quad \eta(t) = -\partial_{\theta(t)} \hat{\Psi}(\cdot), \\ -\partial_{A(t)} \hat{\Psi}(\cdot) \cdot \dot{A}(t) - \frac{1}{\rho_0 \vartheta(t)} \hat{Q}(\sigma) \cdot \nabla\vartheta(t) &\geq 0, \end{aligned}$$

are satisfied for every time $t \in [0, d_p]$.

The internal dissipation function is defined by the expression

$$(2.7) \quad \hat{i}(\sigma) = -\frac{1}{\vartheta(t)} \partial_{A(t)} \hat{\Psi}(\cdot) \cdot \hat{A}(\sigma).$$

The expression (2.7) which defines the value of the internal dissipation at the intrinsic state σ shows that full information given in the method of preparation, i.e. the set of internal state variables $A(t)$ and the evolution equation for these variables $\dot{A}(t) = \hat{A}(\sigma)$, essentially determines the internal dissipation for this intrinsic state σ .

3. Coupling of dissipative mechanisms

In the case of an inelastic material there are two main reasons for internal dissipation. First, the pure viscous effects and second the visco-plastic flow.

To make our reasoning clear let us consider two main physical mechanisms responsible

for inelastic effects, namely the internal friction mechanism and the thermo-activated mechanism (or the damping mechanism).

To describe both inelastic effects simultaneously two groups of internal state variables are introduced⁽³⁾ i.e.

$$(3.1) \quad A(t) = (\alpha(t), \omega(t))$$

for which the following initial value problem

$$(3.2) \quad \begin{aligned} \dot{\alpha}(t) &= \mathfrak{U}(\sigma) = \mathfrak{U}(P(t), \alpha(t), \omega(t)), & \alpha(0) &= \alpha_0, \\ \dot{\omega}(t) &= \mathfrak{Q}(\sigma) = \mathfrak{Q}(P(t), \alpha(t), \omega(t)), & \omega(0) &= \omega_0, \end{aligned}$$

is postulated.

It is assumed that the first group of internal state variables $\alpha(t)$ describes the dissipation effects of the internal friction mechanism, and the second $\omega(t)$ the effects of the thermo-activated mechanism (or the damping mechanism).

It is noteworthy that two groups of internal state variables $\alpha(t)$ and $\omega(t)$ introduced by Eqs. (3.1) and (3.2) can describe coupling between dissipative mechanisms.

4. Particular example of coupling effects

In many cases of practical interest we can find a formalism in which the internal state variables describing effects of the first mechanism are eventually eliminated. We can achieve this in a rather simple manner if the first mechanism is described by the linear evolution equations⁽⁴⁾.

If we consider again the internal friction mechanism and the thermo-activated mechanism (or the damping mechanism) as two main dissipative mechanisms, then we can show a very good example of such a procedure.

It is a very well-known fact that the internal friction mechanism leads to visco-elastic effects and is described by the linear evolution equations. On the other hand the thermo-activated mechanism (or the damping mechanism) is responsible for elastic-viscoplastic effects. The internal state variables introduced to describe these effects are governed by the nonlinear evolution equations.

Let us assume that the evolution equations (3.2) have the form

$$(4.1) \quad \begin{aligned} \dot{\alpha}(t) &= \mathfrak{U}_1[\alpha(t)] - \mathfrak{U}_2(P(t), \omega(t)), \\ \dot{\omega}(t) &= \mathfrak{Q}_1(P(t), \omega(t)) + \mathfrak{Q}_2(P(t), \omega(t))[\alpha(t)], \end{aligned}$$

where \mathfrak{U}_1 is a constant matrix, \mathfrak{U}_2 , \mathfrak{Q}_1 and \mathfrak{Q}_2 may depend in an arbitrary manner on $P(t)$ and $\omega(t)$.

Equation (4.1)₁ possesses the formal solution

$$(4.2) \quad \alpha(t) = \exp(\mathfrak{U}_1 t) \alpha_0 - \int_0^t \exp[\mathfrak{U}_1(t-\tau)] \mathfrak{U}_2(P(\tau), \omega(\tau)) d\tau$$

⁽³⁾ This idea was first introduced by the author in the paper [17].

⁽⁴⁾ For statistical justification of this procedure see R. ZWANZIG [27] and H. HAKEN [7].

which may be transformed by partial integration into

$$(4.3) \quad \alpha(t) = \mathfrak{M}_1^{-1} \mathfrak{M}_2(P(t), \omega(t)) + \exp(\mathfrak{M}_1 t) [\alpha_0 - \mathfrak{M}_1^{-1} \mathfrak{M}_2(P(0), \omega(0))] \\ - \int_0^t \exp[\mathfrak{M}_1(t-\tau)] \mathfrak{M}_1^{-1} \{ \partial_{P(\tau)} \mathfrak{M}_2(P(\tau), \omega(\tau)) \cdot \dot{P}(\tau) + \partial_{\omega(\tau)} \mathfrak{M}_2(P(\tau), \omega(\tau)) \cdot \dot{\omega}(\tau) \} d\tau.$$

Inserting the result (4.3) into Eq. (4.1)₂ yields the desired evolution equation

$$(4.4) \quad \dot{\omega}(t) = \Omega_1(P(t), \omega(t)) + \Omega_2(P(t), \omega(t)) \mathfrak{M}_1^{-1} \mathfrak{M}_2(P(t), \omega(t)) \\ - \int_0^t \Omega_2(P(t), \omega(t)) \exp[\mathfrak{M}_1(t-\tau)] \mathfrak{M}_1^{-1} \{ \partial_{P(\tau)} \mathfrak{M}_2(P(\tau), \omega(\tau)) \cdot \dot{P}(\tau) \\ + \partial_{\omega(\tau)} \mathfrak{M}_2(P(\tau), \omega(\tau)) \cdot \dot{\omega}(\tau) \} d\tau \\ + \Omega_2(P(t), \omega(t)) \exp(\mathfrak{M}_1 t) [\alpha_0 - \mathfrak{M}_1^{-1} \mathfrak{M}_2(P(0), \omega(0))].$$

The internal state variables of the first group enter into the integro-differential equation (4.4) only through their initial values α_0 .

We can obtain an equivalent result by inserting the formal solution (4.2) directly into the evolution equation (4.1)₂. The result is as follows.

$$(4.5) \quad \dot{\omega}(t) = \Omega_1(P(t), \omega(t)) + \Omega_2(P(t), \omega(t)) \{ \exp(\mathfrak{M}_1 t) \alpha_0 \\ - \int_0^t \exp[\mathfrak{M}_1(t-\tau)] \mathfrak{M}_2(P(\tau), \omega(\tau)) d\tau \}, \quad \omega(0) = \omega_0.$$

The initial value problem (4.5) is equivalent to the Volterra integral equation

$$(4.6) \quad \omega(t) = \omega_0 + \int_0^t \Omega_1(P(\tau), \omega(\tau)) d\tau + \int_0^t \Omega_2(P(\xi), \omega(\xi)) \{ \exp(\mathfrak{M}_1 \xi) \alpha_0 \\ - \int_0^\xi \exp[\mathfrak{M}_1(\xi-\tau)] \mathfrak{M}_2(P(\tau), \omega(\tau)) d\tau \} d\xi.$$

The result in the form of the integro-differential equation (4.4) has more direct interpretation than the evolution equation (4.5) or than the Volterra integral equation (4.6).

All terms, except the first, on the right hand side in the integro-differential equation (4.4) reflect influences of the first mechanism on the evolution of the internal state variables of the second group. The second term describes an instantaneous interaction, the integral describes a general retarded interaction and the last term may be interpreted as a fluctuation interaction. All these terms represent the coupling effects between the two dissipative mechanisms considered.

The result obtained in this section is of great importance to us. To describe interactions of both mechanisms it is sufficient to introduce only one group of the internal state variables, namely ω , and to use the modified evolution equation (4.4) or Eq. (4.6). So, in what follows we can assume that the intrinsic state is given by

$$(4.7) \quad \sigma^* = (P(t), \omega(t)).$$

5. Physical foundations

The effects of temperature and strain rate on the plastic flow behaviour of metals have been successfully rationalized in terms of the dynamics of dislocations.

To make our reasoning clear let us consider the movement of a dislocation through the rows of barriers. The moving dislocation is dissipated energy mainly due to two sources. The first is connected with the overcoming of obstacles which involves a thermally activated process and the second with the interactions of dislocation with lattice thermal vibrations (phonon drag) or with electrons (electron viscosity) or are based on relaxation effects in the dislocation core (glide-plane viscosity).

For that reason the theoretical mechanisms governing the velocity of dislocation in relatively pure materials can be divided into two groups, i.e., mechanisms which do and do not involve a thermal activation process.

According to that explanation it is convenient to divide the strain rate sensitivity behaviour of metals into two regions⁽⁵⁾ (cf. Fig. 1):

- the thermally activated region (Region II in Fig. 1) and
- the phonon damping region (Region IV in Fig. 1).

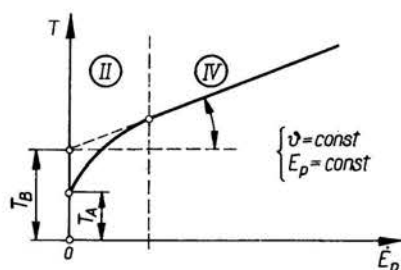


FIG. 1. Two dissipative regions II and IV for stress versus strain rate curve at constant strain and temperature.

The flow stress T can be represented by the equation

$$(5.1) \quad T = T_A + T^* + T_D,$$

where T_A is the athermal component of stress, T^* is the thermally-activated component of stress, and T_D is the stress attributed to phonon damping.

If a dislocation is moving through the rows of barriers, then its velocity can be determined by the expression⁽⁶⁾

$$(5.2) \quad v = \mathcal{A}L^{-1}/(t_s + t_B)$$

where $\mathcal{A}L^{-1}$ is the average distance of dislocation movement after each thermal activation, t_s is the time a dislocation spent at the obstacle and t_B is the time of travelling between the barriers.

⁽⁵⁾ We use here the denotation of the dissipative regions as proposed by A. R. ROSENFELD and G. T. HAHN [24]. They have shown that in the temperature-strain rate spectrum of plain carbon steel four regions can be considered which reflect different mechanisms of plastic deformations.

⁽⁶⁾ See a theoretical analysis presented by A. KUMAR and R. G. KUMBLE [11] and also a very recent paper by C. TEODOSIU and F. SIDOROFF [26].

The plastic strain rate \dot{E}_p is given by the Orowan relation

$$(5.3) \quad \dot{E}_p = \varrho_M \mathbf{b} v,$$

where ϱ_M is the mobile dislocation density, and \mathbf{b} denotes the Burger's vector and v is the average dislocation velocity.

The average velocity of a dislocation, damped by the phonons between thermally activated events which are computed by the relation (5.2), takes the form of the expression

$$(5.4) \quad v = \mathcal{A} L^{-1} [\nu^{-1} \exp(U/k\vartheta) + \mathcal{A} B L^{-1} / (T - T_B) \mathbf{b}]^{-1},$$

where ν is the vibration frequency of the dislocation segment, $U(T^*)$ is the activation energy, $k\vartheta$ is the Boltzmann constant times the absolute temperature, B denotes the dislocation drag coefficient, and T_B is attributed to the stress needed to overcome the forest dislocation barriers to the dislocation motion (cf. Fig. 1).

After inserting the result (5.4) into Eq. (5.3) we obtain the desired equation for the plastic strain rate

$$(5.5) \quad \dot{E}_p = \varrho_M \mathbf{b} \mathcal{A} L^{-1} [\nu^{-1} \exp(U(T^*)/k\vartheta) + \mathcal{A} B L^{-1} / (T - T_B) \mathbf{b}]^{-1}.$$

Equation (5.5) is the evolution equation for the plastic strain E_p in the case of simultaneous interaction of the thermally-activated mechanism and the phonon damping mechanism.

It will be useful to consider two limit cases as follows.

a) If the time t_B taken by the dislocation to travel between the barriers in a viscous phonon medium is negligible when compared with the time t_s spent at the obstacle, then we can focus our attention on the analysis of the thermally-activated process⁽⁷⁾ (Region II in Fig. 1). The dislocation velocity, Eq. (5.2), can be approximated by the expression

$$(5.6) \quad v = \frac{\mathcal{A} L^{-1}}{t_s}.$$

The evolution equation for the plastic strain E_p takes now the form

$$(5.7) \quad \dot{E}_p = (\varrho_M/L) \mathcal{A} \mathbf{b} \nu \exp[-U(T^*)/k\vartheta].$$

Since the flow stress T for this case is given by the relation

$$(5.8) \quad T = T_A + T^*,$$

the activation energy U can be assumed as the nonlinear function of the overstress $\frac{T}{T_A} - 1$,

i.e.

$$(5.9) \quad U(T^*) = \varphi \left[a \left(\frac{T}{T_A} - 1 \right) \right],$$

where a is constant.

If we additionally introduce the denotation

$$(5.10) \quad \gamma_1 = (\varrho_M/L) \mathbf{b} \mathcal{A} \nu,$$

⁽⁷⁾ The mechanism for overcoming the dislocation forest which may appear in crystals of different metals in various temperature ranges has been developed theoretically by A. SEEGER [25].

then the evolution equation (5.7) takes the very well-known form

$$(5.11) \quad \dot{E}_p = \gamma_1 \exp \left\{ -\varphi \left[a \left(\frac{T}{T_A} - 1 \right) \right] / k\vartheta \right\}.$$

In the description proposed we have three parameters E_p , γ_1 and T_A which can be treated as the internal state variables (cf. here the results of the paper by PERZYNA and WOJNO [23]).

b) With increasing dislocation velocities, the ratio t_B/t_s increases and at a high enough stress or in a perfect crystal the velocity is only governed by the phonon damping mechanism⁽⁸⁾ (cf. Region IV in Fig. 1). At very high strain rates the applied stress is high enough to overcome instantaneously the dislocation barriers without any aid from thermal fluctuations. This is true for the flow stress $T > T_B$, where T_B is attributed to the stress needed to overcome the forest dislocation barriers to the dislocation motion and is called the back stress.

In this region the dislocation velocity, Eq. (5.2), can be approximated by the expression

$$(5.12) \quad v = \frac{\mathcal{A}L^{-1}}{t_B}$$

and the evolution equation for the plastic strain takes the form⁽⁹⁾

$$(5.13) \quad \dot{E}_p = \frac{\varrho_M \mathbf{b}^2}{B} (T - T_B).$$

The flow stress T consists of two terms

$$(5.14) \quad T = T_B + T_D, \quad \text{where} \quad T_B = T_A + T_0^*.$$

If we introduce the denotation

$$(5.15) \quad \gamma_2 = \frac{\varrho_M \mathbf{b}^2 T_B}{B}$$

then the evolution equation (5.13) has a simpler form

$$(5.16) \quad \dot{E}_p = \gamma_2 \left(\frac{T}{T_B} - 1 \right).$$

In the description discussed here we have, as in the previous case, three parameters, namely E_p , γ_2 and T_B which can be treated as the internal state variables.

When a crystal is set into vibration, the vibrations decay even if there is no loss of energy from the crystal to its surroundings. The cause of the decay is called internal friction. Internal friction may arise by a number of mechanisms⁽¹⁰⁾. In the linear approxima-

⁽⁸⁾ The phonon viscosity theory has been developed by W. P. MASON [14]. For a thorough theoretical discussion of damping mechanisms see F. R. N. NABARRO [15] and J. A. GORMAN, D. S. WOOD and T. VREELAND [6].

⁽⁹⁾ The dislocation drag coefficient B can be interpreted as a generalized damping constant for phonon viscosity and phonon scattering mechanisms, i.e., $B = B_{pv} + B_{ps}$, cf. Ref. [20]. The ratio B/ϱ_M can be obtained from the slope of the linear portion of the stress against the strain rate curve, and from this ratio the value of B can be calculated if a value of ϱ_M is assumed.

⁽¹⁰⁾ Comprehensive review of the internal friction mechanisms can be found in the treatise by F. R. N. NABARRO [15].

tion the theory of each of these mechanisms leads to the known Boltzmann constitutive equations of viscoelasticity⁽¹¹⁾.

As it has been suggested by F.R.N. NABARRO [15] the presence of dislocations in the crystal may increase the internal friction in three general ways. Firstly, the dislocations act as sources of internal stresses and contribute to the thermoelastic damping. Secondly, the dislocations may move under the action of an applied stress, thereby producing a modulus defect. If the motion of the dislocations is resisted by one of the mechanisms discussed in this Section their displacement is not in phase with the applied stress, and the modulus defect has an imaginary component which corresponds to an internal friction. Thirdly, the motion of the dislocations may be hindered by localized obstacles. A dislocation acquires energy in order to overcome an obstacle, and this energy is dissipated when the obstacle has been passed by a mechanism which contribute to the mechanisms of dislocation damping.

6. One-dimensional examples of coupling phenomena

It will be useful for future applications to consider approximations of the description previously presented.

a) For some metals (e.g. iron, mild steel and titanium) for medium strain rates the thermally-activated mechanism becomes dominant. In this particular case we can extend Region II to be valid for the entire range of strain rate and temperature changes, Fig. 2.

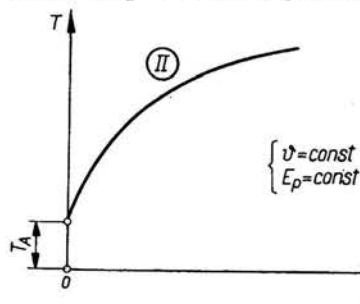


FIG. 2. Approximation by the thermally activated mechanism in the entire range of strain rate (Region II).

Basing on the results obtained in the previous section we introduce the internal state variables as follows:

$$(6.1) \quad \omega(t) = [E_p(t), \gamma(t), \kappa(t)],$$

where for the one-dimensional case $E_p(t)$ is a scalar value interpreted as the inelastic strain, $\gamma(t)$ is the viscosity parameter defined by Eq. (5.10), and $\kappa(t)$ is interpreted as the athermal stress $T_A(t)$.

For the internal state variables introduced we postulate the evolution equations in the form

$$(6.2) \quad \begin{aligned} \dot{E}_p(t) &= \gamma(t) \Phi \left[\frac{T(t)}{\kappa(t)} - 1 \right], \\ \dot{\gamma}(t) &= \hat{\Gamma}(\sigma^*) \dot{E}_p(t), \\ \dot{\kappa}(t) &= \hat{K}(\sigma^*) \dot{E}_p(t), \end{aligned}$$

⁽¹¹⁾ Statistical mechanics of viscoelasticity based on the internal friction mechanisms has been developed by G. P. DE VAULT and J. A. MCLENNAN [3, 4] and by H. J. MARIS [13].

where Φ is a new dimensionless function of the overstress, $\hat{\Gamma}(\sigma^*)$ and $\hat{K}(\sigma^*)$ are functions of the intrinsic state

$$(6.3) \quad \sigma^* = (E(t), \vartheta(t), \omega(t)).$$

It is intended that all these functions may be chosen to represent the results of tests on the dynamic behaviour of metals.

To be more precise we shall consider the example of experimental data for pure titanium. J. E. LAWSON and T. NICHOLAS [12] have performed series of experiments on pure titanium in which they obtained the linear dependence of shear flow stress on log (strain rate) for a very large range of strain rate changes (from 10^{-4} to 10^4 s $^{-1}$). These data are shown in Figs. 3 and 4.

To represent these experimental data we assume the evolution equation for the internal state variable E_p to be as follows

$$(6.4) \quad \dot{E}_p(t) = \hat{\gamma}(E_p(t), \vartheta(t)) \left\{ \exp \left[\frac{T(t)}{\hat{f}(E_p(t), \vartheta(t))} - 1 \right] - 1 \right\}$$

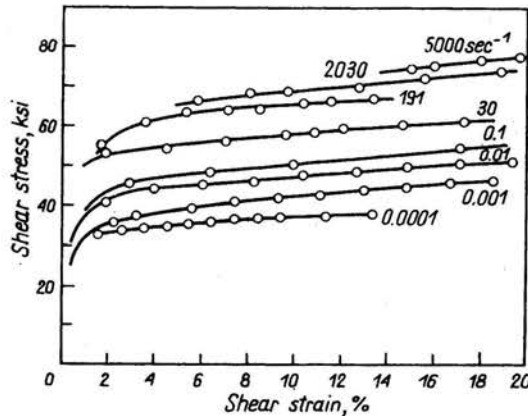


FIG. 3. Shear stress-strain curves at constant strain rate for commercially pure titanium (After J. E. LAWSON and T. NICHOLAS [12]).

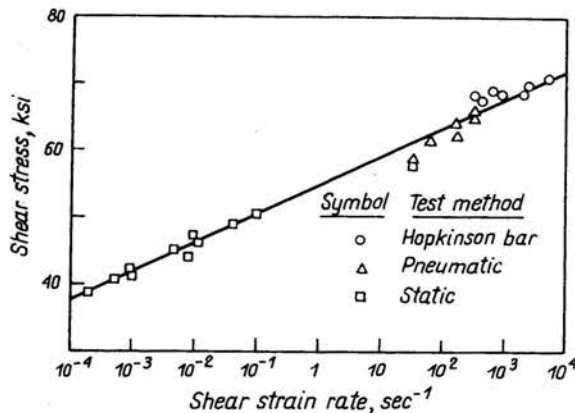


FIG. 4. Stress versus log (strain rate) at 10 per cent strain for commercially pure titanium (After J. E. LAWSON and T. NICHOLAS [12]).

in which $\Phi(\cdot) \equiv \exp(\cdot) - 1$ and the internal state variables $\gamma(t)$ and $\varkappa(t)$ are postulated in the form

$$(6.5) \quad \begin{aligned} \gamma(t) &= \hat{\gamma}(E_p(t), \vartheta(t), \\ \varkappa(t) &= \hat{f}(E_p(t), \vartheta(t)), \end{aligned}$$

where the function \hat{f} represents a description of the static curve for titanium. For an isothermal case this function can be taken as the approximation of the curve obtained for $\vartheta = \text{const}$ and $\dot{E}_p(t) = \text{const} = 10^{-4} \text{ s}^{-1}$ (cf. Fig. 4).

Basing on similar data for nonisothermal tests and for different plastic strains $E_p(t)$ we can determine the function $\hat{\gamma}$.

Let us describe now the coupling effects between the internal friction and thermally-activated mechanisms for the one-dimensional case.

To do this we assume evolution equations for the internal state variables of the second group, i.e. for $\omega(t) = [E_p(t), \gamma(t), \varkappa(t)]$ in the following form:

$$(6.6) \quad \dot{\omega}(t) = \Omega_1(\sigma^*) + \Omega_2(\sigma^*) [\alpha(t)]$$

with the initial value $\omega(0) = \omega_0$, where

$$(6.7) \quad \begin{aligned} \Omega_1(\sigma^*) &= \gamma(t) \Phi \left(\frac{T(t)}{\varkappa(t)} - 1 \right) [1, \hat{\Gamma}(\sigma^*), \hat{K}(\sigma^*)], \\ \Omega_2(\sigma^*) &= [\hat{h}(\sigma^*), \hat{m}(\sigma^*), 0]. \end{aligned}$$

The functions $\hat{h}(\sigma^*)$ and $\hat{m}(\sigma^*)$ describe the influences of the internal friction mechanism on the changes of the inelastic strain E_p and of the viscosity parameter γ . It is postulated that there is no influence of the internal friction mechanism on the athermal stress $\varkappa \equiv T_A$.

The evolution equations for the first group of the internal state variables (i.e. for $\alpha = [\alpha_1, \dots, \alpha_n]$, n is finite) are postulated in the form of Eq. (4.1).

After eliminating the internal state variables of the first group we obtain the Volterra integral equation (cf. with the result (4.6)):

$$(6.8) \quad \omega(t) = \omega_0 + \int_0^t \Omega_1(\sigma^*(\tau)) d\tau + \int_0^t \Omega_2(\sigma^*(\xi)) \left\{ \exp[\mathfrak{A}_1 \xi] \alpha_0 - \int_0^\xi \exp[\mathfrak{A}_1(\xi - \tau)] \mathfrak{A}_2(\sigma^*(\tau)) d\tau \right\} d\xi.$$

If we use the experimental data of J. L. LAWSON and T. NICHOLAS for titanium again then the material functions Ω_1 and Ω_2 in the integral equation (6.8) take the form

$$(6.9) \quad \begin{aligned} \Omega_1(\sigma^*) &= \hat{\gamma}(E_p(t), \vartheta(t)) \left\{ \exp \left[\frac{T(t)}{\hat{f}(E_p(t), \vartheta(t))} - 1 \right] - 1 \right\}, \\ \Omega_2(\sigma^*) &= \hat{h}(\sigma^*). \end{aligned}$$

The functions left to be determined are the influence functions $\mathfrak{A}_2(\sigma^*)$ and $\hat{h}(\sigma^*)$.

b) Some recent investigations on metals have shown that at very high strain rates the phonon damping mechanism becomes most influential. For some metals (e.g. alu-

minium and copper) this mechanism can approximate the plastic flow phenomena in the entire range of strain rates changes. The situation is schematically shown in Fig. 5. This approximation has been suggested for aluminium single crystals by W. G. FERGUSON, A. KUMAR and J. E. DORN [5], A. KUMAR, F. E. HAUSER and J. E. DORN [10] and J. A. GORMAN, D. S. WOOD and T. VREELAND [6], for polycrystalline aluminium by F. E. HAUSER, J. A. SIMMONS and J. E. DORN [8], C. H. KARNES and E. A. RIPPERGER [9] and A. KUMAR, F. E. HAUSER and J. E. DORN [10] and for copper by A. KUMAR and R. G. KUMBLE [11].

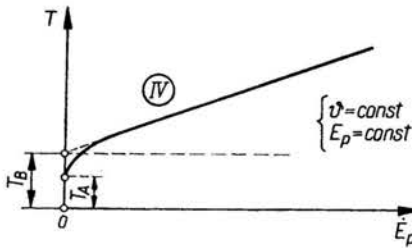


FIG. 5. Approximation by the phonon damping mechanism in the entire range of strain rate (Region IV).

The flow stress of aluminium crystals at different strain rates and strains at temperature 10 K, 77 K, 300 K and 500 K obtained by A. KUMAR, F. E. HAUSER and J. E. DORN [10] is shown in Fig. 6. It is apparent that the linear relationship between stress and strain rate does continue to prevail at high strain rates without any deviation even at a 20 per cent strain. The linear portion of the stress versus strain rate curve when extrapolated intersects the stress axis at T_B . This has been defined as a back stress. The results plotted in Fig. 6 clearly show that the back stress T_B for aluminium single crystals is a function of strain or dislocation density and is independent of temperature within the accuracy of the measurements.

Data from the paper by F. E. HAUSER, J. A. SIMMONS and J. E. DORN [8] for polycrystalline aluminium which were originally plotted on a semilogarithmic scale are shown in Fig. 7 on a linear scale. Again it is observed that the magnitude of T_B increases with an increase in strain, Fig. 8. However, for the polycrystalline aluminium the back stress T_B does depend on temperature, namely T_B decreases with an increase in temperature, Fig. 9. At slower strain rates the damping term is negligible and the strain rate sensitivity

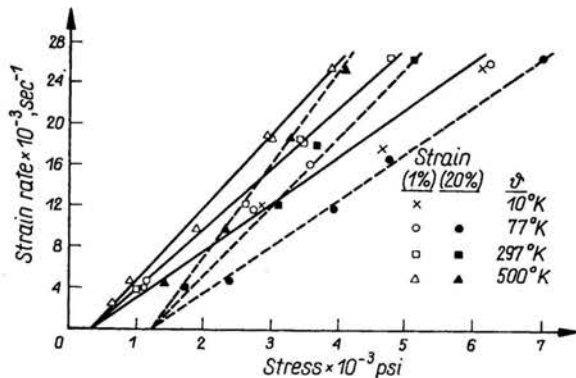


FIG. 6. The strain rate dependence of the flow stress of aluminium single crystals deformed by $(1, 1, 1)$ $\langle 1, \bar{1}, 0 \rangle$ slip (After A. KUMAR, F. E. HAUSER and J. E. DORN [10]).

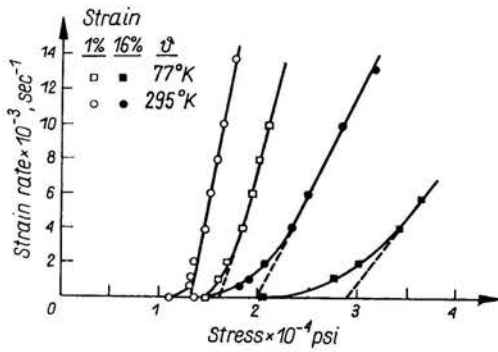


FIG. 7. The strain rate dependence of the yield stress for polycrystalline aluminium. The data of F. E. HAUSER, J. A. SIMMONS and J. E. DORN [8] are plotted on a linear scale (After A. KUMAR, F. E. HAUSER and J. E. DORN [10]).

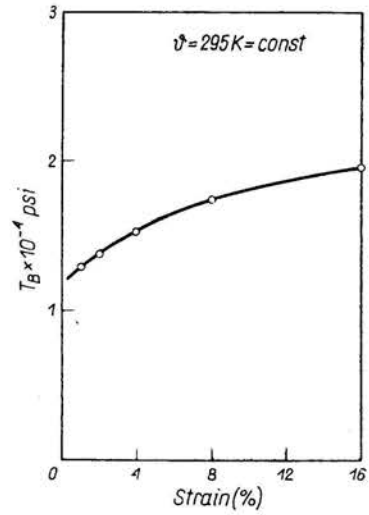


FIG. 8. The dependence of the back stress T_B on plastic strain for aluminium at constant temperature $\vartheta = 295$ K. Based on the experimental data of F. E. HAUSER, J. A. SIMMONS and J. E. DORN [8].

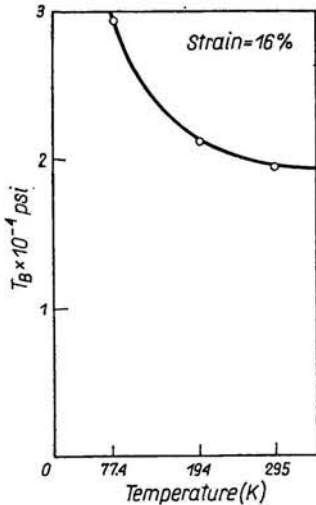


FIG. 9. The dependence of the back stress T_B on temperature for aluminium at constant plastic strain $E_p = 16\%$. Based on the experimental data of F. E. HAUSER, J. A. SIMMONS and J. E. DORN [8].

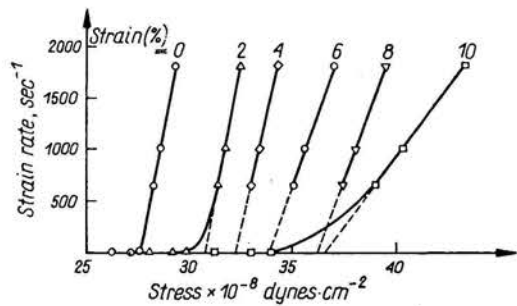


FIG. 10. The linear dependence of flow stress on strain rate at high strain rates for copper at different plastic strains and at 300 K (After A. KUMAR and R. G. KUMBLE [11]).

in Eq. (5.1) arises from the thermally-activated component of the stress T^* . At very high strain rates there is no assistance from the thermal fluctuations, and the strain rate sensitivity is attributed to the third term in Eq. (5.1) which is a dislocation damping term T_D .

The mechanical behaviour of copper at strain rates from 10^{-3} to 10^3 s^{-1} at temperature 300 K, 420 K and 590 K was investigated by A. KUMAR and R. G. KUMBLE [11]. Data obtained by these authors have been plotted on a linear scale in Figs. 10, 11 and 12. In these plots it is observed that the strain rate behaviour of copper can be divided into two regions. Below 10 s^{-1} the mobile dislocations are thermally activated over the forest dislocation barriers and above 10^2 s^{-1} the dislocation motion is viscous drag limited.

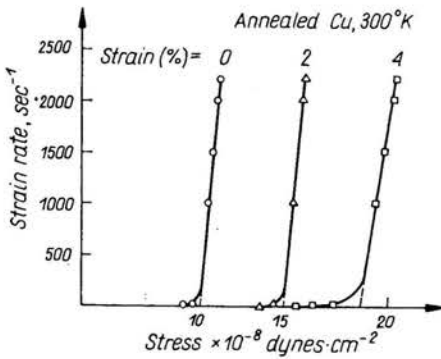


FIG. 11. The linear dependence of flow stress on strain rate at high strain rates for annealed copper at plastic strain 0.2 and 4% and at 300 K (After A. KUMAR and R. G. KUMBLE [11]).

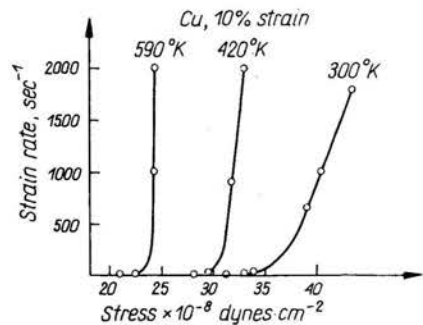


FIG. 12. The linear dependence of the flow stress for copper at high strain rates at temperature 590, 420 and 300 K and at 10% plastic strain (After A. KUMAR and R. G. KUMBLE [11]).

The flow stress is a linear function of strain rate at high strain rates for all the states at 300, 420 and 590 K and can be represented by Eq. (5.16). The magnitude of T_B for copper and its variation with strain is wholly consistent with the concept that T_B is the stress needed to overcome the usual barriers to the dislocation motion. The authors of the paper concluded that in the case of copper these barriers are forest dislocations. The magnitude of T_B in Fig. 12 decreases with increasing temperature for a strain-hardened state. This has been attributed to the recovery of the dislocation structure.

The authors observed also that the mobile dislocation density in the viscous phonon damping region is independent of strain up to 6% and decreases slightly with a further increase in strain. The mobile dislocation density increases with increasing temperature.

This last observation is of great importance to our phenomenological description based on internal state variables because it concerns the viscosity parameter γ_2 defined by the relation (5.15).

A discussion of the experimental data for aluminium single crystals, polycrystalline aluminium and copper clearly shows that the second approximation proposed is justified.

To describe this simplified case let us introduce the internal state variables as postulated by Eq. (6.1) and interpret $\gamma(t) = \gamma_2(t)$ and $\kappa(t) = T_B(t)$.

Basing on the experimental results discussed and the physical propositions given in Sect. 5b we postulate the evolution equations for the internal state variables as follows:

$$(6.10) \quad \begin{aligned} \dot{E}_p(t) &= \gamma(t) \left[\frac{T(t)}{\kappa(t)} - 1 \right], \\ \dot{\gamma}(t) &= \bar{\Gamma}(\sigma^*) \dot{E}_p(t), \\ \dot{\kappa}(t) &= \bar{K}(\sigma^*) \dot{E}_p(t), \end{aligned}$$

where $\bar{\Gamma}$ and \bar{K} are new functions of the intrinsic state σ^* .

Our description will be simpler and more useful if, basing on the experimental data discussed, we can determine directly the material functions (cf. Figs. 8 and 9)

$$(6.11) \quad \begin{aligned} \gamma(t) &= \bar{\gamma}(E_p(t), \vartheta(t)), \\ \kappa(t) &= \bar{\kappa}(E_p(t), \vartheta(t)). \end{aligned}$$

If that is the case the phenomenological theory is straightforward and involves only one internal state variable $E_p(t)$ with the evolution equation in the form

$$(6.12) \quad \dot{E}(t) = \bar{\gamma}(E_p(t), \vartheta(t)) \left[\frac{T(t)}{\bar{\kappa}(E_p(t), \vartheta(t))} - 1 \right].$$

The coupling between the internal friction and the phonon damping mechanisms is described by the integral equation (6.8) in which the functions Ω_1 and Ω_2 are defined as follows:

$$(6.13) \quad \begin{aligned} \Omega_1(\sigma^*) &= \gamma(t) \left[\frac{T(t)}{\kappa(t)} - 1 \right] [1, \bar{\Gamma}(\sigma^*), \bar{K}(\sigma^*)], \\ \Omega_2(\sigma^*) &= [\bar{h}(\sigma^*), \bar{m}(\sigma^*), 0], \end{aligned}$$

where $\bar{h}(\sigma^*)$ and $\bar{m}(\sigma^*)$ are new influence functions.

When $\gamma(t)$ and $\kappa(t)$ are given by the relations (6.11), the functions Ω_1 and Ω_2 take the form

$$(6.14) \quad \begin{aligned} \Omega_1(\sigma^*) &= \bar{\gamma}(E(t), \vartheta(t)) \left[\frac{T(t)}{\bar{\kappa}(E(t), \vartheta(t))} - 1 \right], \\ \Omega_2(\sigma^*) &= \bar{h}(\sigma^*). \end{aligned}$$

A description of this simple case is reduced to the Volterra integral equation

$$(6.15) \quad \begin{aligned} E_p(t) = E_p^0 + \int_0^t \bar{\gamma}(E_p(\tau), \vartheta(\tau)) \left[\frac{T(\tau)}{\bar{\kappa}(E_p(\tau), \vartheta(\tau))} - 1 \right] d\tau \\ + \int_0^t \bar{h}(E(\xi), \vartheta(\xi), E_p(\xi)) \left\{ \exp(\mathfrak{A}_1 \xi) \alpha_0 \right. \\ \left. - \int_0^\xi \exp[\mathfrak{A}_1(\xi - \tau)] \mathfrak{A}_2(E(\tau), \vartheta(\tau), E_p(\tau)) d\tau \right\} d\xi, \end{aligned}$$

where E_p^0 denotes the initial value of the inelastic strain $E_p(t)$, i.e. $E_p(0) = E_p^0$.

The crucial point of the theory describing the influence of the internal friction mechanism on the phonon damping plastic flow is the determination of the functions $\mathfrak{U}_2(\sigma^*)$ and $\bar{h}(\sigma^*)$. Particular functions $\mathfrak{U}_2(\sigma^*)$ and $\bar{h}(\sigma^*)$ are assumed the influence of the second integral on the solution of the Volterra integral equation (6.15) is settled.

It is noteworthy that the realistic determination of the functions $\mathfrak{U}_2(\sigma^*)$ and $\bar{h}(\sigma^*)$ is difficult. The reason for that is the lack of experimental tests investigating the influence discussed.

7. Three-dimensional theory

To generalize the description of the coupling effects between the internal friction and plastic flow mechanisms for the case of combined states of stress and strain we postulate the internal state variables as follows:

$$(7.1) \quad \omega(t) = [E_p(t), \gamma(t), \varkappa(t)],$$

where $E_p(t)$ denotes now the inelastic strain tensor, $\gamma(t)$ is the viscosity parameter and $\varkappa(t)$ is interpreted as the work-hardening parameter.

As it was shown in the previous sections we can approximate the rate sensitive plastic flow by the thermally-activated mechanism or by the phonon damping mechanism.

In both cases the description is given by the constitutive equations

$$(7.2) \quad Z(t) = \hat{\mathbf{S}}(\sigma^*) = \hat{\mathbf{S}}(P(t), \omega(t)), \quad \hat{\mathbf{S}} \equiv \{\hat{\Psi}, \hat{\mathbf{N}}, \hat{\mathbf{T}}, \hat{\mathbf{Q}}\}$$

with the restrictions (cf. Eqs. (2.6))

$$(7.3) \quad \begin{aligned} \partial_{\nabla\theta(t)}\hat{\Psi}(\cdot) &= 0, \quad \mathbf{T}(t) = 2\rho_0 \partial_{C(t)}\hat{\Psi}(\cdot), \quad \eta(t) = -\partial_{\theta(t)}\hat{\Psi}(\cdot), \\ -\partial_{\omega(t)}\hat{\Psi}(\cdot) \cdot \dot{\omega}(t) - \frac{1}{\rho_0\theta(t)} \hat{\mathbf{Q}}(\sigma^*) \cdot \nabla\theta(t) &\geq 0, \quad t \in [0, d_p] \end{aligned}$$

and by the integro-differential equation (4.4) (or by the Volterra integral equation (4.6)) in which the functions Ω_1 and Ω_2 take the form

$$(7.4) \quad \begin{aligned} \Omega_1(\sigma^*) &= \gamma(t) \left\langle \Phi \left(\frac{f(t)}{\varkappa(t)} - 1 \right) \right\rangle [\partial_{T(t)}f, \text{tr}(\partial_{T(t)}f\hat{\Gamma}(\sigma^*)), \text{tr}(\partial_{T(t)}f\hat{\mathbf{K}}(\sigma^*))], \\ \Omega_2(\sigma^*) &= [\hat{\mathbf{H}}(\sigma^*), \hat{\mathbf{M}}(\sigma^*), 0], \end{aligned}$$

for thermally-activated plastic flow, and

$$(7.5) \quad \begin{aligned} \Omega_1(\sigma^*) &= \gamma(t) \left\langle \frac{f(t)}{\varkappa(t)} - 1 \right\rangle [\partial_{T(t)}f, \text{tr}(\partial_{T(t)}f\bar{\Gamma}(\sigma^*)), \text{tr}(\partial_{T(t)}f\bar{\mathbf{K}}(\sigma^*))], \\ \Omega_2(\sigma^*) &= [\bar{\mathbf{H}}(\sigma^*), \bar{\mathbf{M}}(\sigma^*), 0], \end{aligned}$$

for phonon damping plastic flow, respectively.

In this description we took advantage of a notion of the quasi-static yield criterion for an elastic-viscoplastic material defined as follows⁽¹²⁾:

⁽¹²⁾ The existence of the quasi-static yield criterion is one of the fundamental assumptions of viscoplasticity theory, cf. Refs. [16-20].

$$(7.6) \quad \mathcal{F}(\sigma^*) = \frac{f(t)}{\kappa(t)} - 1,$$

where

$$(7.7) \quad f(t) = f(T(t), E_p(t), \vartheta(t)),$$

and we introduce the symbol $\langle [] \rangle$ according to the definition

$$\langle [] \rangle = \begin{cases} 0 & \text{if } f(t) \leq \kappa(t), \\ [] & \text{if } f(t) > \kappa(t). \end{cases}$$

The material functions $\hat{\Gamma}, \hat{\mathbf{K}}, \hat{\mathbf{H}}, \hat{\mathbf{M}}$ and $\bar{\Gamma}, \bar{\mathbf{K}}, \bar{\mathbf{H}}, \bar{\mathbf{M}}$ are understood as a simple generalization for the three-dimensional case of the functions $\hat{\Gamma}, \hat{\mathbf{K}}, \hat{h}, \hat{m}$ and $\bar{\Gamma}, \bar{\mathbf{K}}, \bar{h}, \bar{m}$, respectively, defined and interpreted in the one-dimensional consideration in Sect. 6.

In the case when material under consideration has two very distinct regions of viscoplastic flow, namely the thermally-activated region (say, from strain rate 10^{-4} s^{-1} to 10^2 s^{-1}) and the phonon damping region (say, from strain rate 10^2 s^{-1} to 10^5 s^{-1}) we can apply the idea presented in Refs. [18] and [23] to describe the viscoplastic response of a material by unified constitutive equations valid in the entire range of strain rate and temperature changes.

It is also very easy to imagine the generalized description taking into simultaneous account the coupling effects between internal friction, thermally-activated and phonon damping mechanisms.

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Received February 2, 1977.