Thus $\nu_1, \nu_2, \nu_3, \ldots$ all satisfy the equation

$$\nu(p)\,\nu(q) = \nu(pq),$$

p and q being prime to each other.

§ 11. In the case of the formulæ considered in §§ 1–8, we have

$$\alpha_n = \frac{B_n}{(2n)!}\, 2^{2n},$$

and

$$\phi(x) = x\sum_1^\infty \frac{1}{2^n} \tan \frac{x}{2^n}.$$

The fact that $\phi(x)$ is itself a series of terms in which the denominators are the successive powers of 2 is the cause of the distinction which occurs in the formulæ of § 5 between 2 and the other prime factors of n. In the formation of the functions θ the exponent of 2 gives rise to a factorial which is one order higher than the factorials depending upon the exponents of the other primes.

EXPANSIONS OF K, I, G, E IN POWERS OF $k'^2 - k^2$.

By J. W. L. Glaisher.

THE object of this note is to give the expansion of K in ascending powers of $k'^2 - k^2$. I have also added the corresponding expansions of I, G, and E.

Expansion of K, §§ 1, 2.

§ 1. Let h and h' denote k^2 and k'^2 respectively, and let

$$\lambda = h' - h = k'^2 - k^2.$$

If therefore α be the modular angle, so that $k = \sin\alpha$, then $\lambda = \cos 2\alpha$.

We have*

$$\frac{\sqrt{\pi}}{4} K = \int_0^\infty \int_0^\infty e^{-x^4 - y^4 - 2\lambda x^2 y^2} dx\, dy,$$

whence, expanding in powers of λ,

$$\frac{\sqrt{\pi}}{4} K = \int_0^\infty \int_0^\infty e^{-x^4 - y^4} \left\{ 1 - 2\lambda x^2 y^2 + \frac{2^2 \lambda^2}{2!} x^4 y^4 - \frac{2^3 \lambda^3}{3!} x^6 y^6 + \&c. \right\}.$$

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* *Proceedings of the London Mathematical Society*, vol. XIII. p. 92 (1881).

Now
$$\int_0^\infty e^{-x^4}\ dx = \tfrac{1}{4}\Gamma\left(\tfrac{1}{4}\right),$$

$$\int_0^\infty e^{-x^4}x^2 dx = \tfrac{1}{4}\Gamma\left(\tfrac{3}{4}\right),$$

$$\int_0^\infty e^{-x^4}x^4 dx = \tfrac{1}{4}\Gamma\left(\tfrac{5}{4}\right),$$

$$\text{\&c.}\qquad\text{\&c.} ;$$

whence

$$\frac{\sqrt{\pi}}{4}K = \frac{1}{4^2}\Gamma^2\left(\tfrac{1}{4}\right)\left\{1 + \frac{1}{2.4}\lambda^2 + \frac{1^2.5^2}{2.4.6.8}\lambda^4 + \frac{1^2.5^2.9^2}{2.4.6.8.10.12}\lambda^6 + \text{\&c.}\right\}$$
$$- \tfrac{1}{4}\Gamma^2\left(\tfrac{3}{4}\right)\left\{\frac{1}{2}\lambda + \frac{3^2}{2.4.6}\lambda^3 + \frac{3^2.7^2}{2.4.6.8.10}\lambda^5 + \frac{3^2.7^2.11^2}{2.4.6\ldots14} + \text{\&c.}\right\}.$$

Denoting by K° the value of K corresponding to the modulus $\dfrac{1}{\sqrt{2}}$, we know that

$$4\sqrt{\pi}.K^\circ = \Gamma^2\left(\tfrac{1}{4}\right).$$

Also, since $\qquad \Gamma\left(\tfrac{1}{4}\right)\Gamma\left(\tfrac{3}{4}\right) = \sqrt{2}.\pi,$

it follows that $\qquad \dfrac{\pi^{\frac{3}{2}}}{2K^\circ} = \Gamma^2\left(\tfrac{3}{4}\right).$

Thus we find

$$K = K^\circ\left\{1 + \frac{1^2}{2.4}\lambda^2 + \frac{1^2.5^2}{2.4.6.8}\lambda^4 + \frac{1^2.5^2.9^2}{2.4.6.8.10.12}\lambda^6 + \text{\&c.}\right\}$$
$$- \frac{\pi}{2K^\circ}\left\{\frac{1}{2}\lambda + \frac{3^2}{2.4.6}\lambda^3 + \frac{3^2.7^2}{2.4.6.8.10}\lambda^5 + \frac{3^2.7^2.11^2}{2.4.6\ldots14}\lambda^7 + \text{\&c.}\right\}.$$

Since $\qquad 2G^\circ K^\circ = \tfrac{1}{2}\pi,$

we have $\qquad 2G^\circ = \dfrac{\pi}{2K^\circ};$

and we may therefore write the expansion of K in the form

$$K = K^\circ\left\{1 + \frac{1^2}{2.4}\lambda^2 + \frac{1^2.5^2}{2.4.6.8}\lambda^4 + \frac{1^2.5^2.9^2}{2.4.6.8.10.12}\lambda^6 + \text{\&c.}\right\}$$
$$- 2G^\circ\left\{\frac{1}{2}\lambda + \frac{3^2}{2.4.6}\lambda^3 + \frac{3^2.7^2}{2.4.6.8.10}\lambda^5 + \frac{3^2.7^2.11^2}{2.4.6\ldots14}\lambda^7 + \text{\&c.}\right\}.$$

L 2

The quantities K° and G° are of course mere numerical constants, their values to eight places of decimals being

$$K^\circ = 1\cdot85407468,$$
$$G^\circ = 1\cdot17875274.$$

§ 2. By changing the modulus from k to k' we change the sign of λ; we thus find

$$K' + K = 2K^\circ \left\{ 1 + \frac{1^2}{2.4}\lambda^2 + \frac{1^2.5^2}{2.4.6.8}\lambda^4 + \frac{1^2.5^2.9^2}{2.4.6.8.10.12}\lambda^6 + \&c. \right\}$$

$$K' - K = 4G^\circ \left\{ \frac{1}{2}\lambda + \frac{3^2}{2.4.6}\lambda^3 + \frac{3^2.7^2}{2.4.6.8.10}\lambda^5 + \frac{3^2.7^2.11^2}{2.4.6...14}\lambda^7 + \&c. \right\}.$$

Expansion of G, § 3.

§ 3. We may deduce the expansion of G from that of K by means of the formula*

$$G = 2hh'\frac{dK}{dh};$$

for $$h = \tfrac{1}{2}(1 - \lambda), \quad h' = \tfrac{1}{2}(1 + \lambda),$$

whence $$G = -(1 - \lambda^2)\frac{dK}{d\lambda}.$$

We thus find

$$G = \quad G^\circ \left\{ 1 + \frac{1}{2.4}\lambda^2 + \frac{3^2}{2.4.6.8}\lambda^4 + \frac{3^2.7^2}{2.4.6.8.10.12}\lambda^6 + \&c. \right\}$$

$$- \frac{K^\circ}{2}\left\{ \frac{1}{2}\lambda + \frac{1^2}{2.4.6}\lambda^3 + \frac{1^2.5^2}{2.4.6.8.10}\lambda^5 + \frac{1^2.5^2.9^2}{2.4.6...14}\lambda^7 + \&c. \right\}.$$

By changing the modulus from k to k' we deduce also

$$G' + G = 2G^\circ \left\{ 1 + \frac{1}{2.4}\lambda^2 + \frac{3^2}{2.4.6.8}\lambda^4 + \frac{3^2.7^2}{2.4.6.8.10.12}\lambda^6 + \&c. \right\}$$

$$G' - G = K^\circ \left\{ \frac{1}{2}\lambda + \frac{1^2}{2.4.6}\lambda^3 + \frac{1^2.5^2}{2.4.6.8.10}\lambda^5 + \frac{1^2.5^2.9^2}{2.4.6...41}\lambda^7 + \&c. \right\}.$$

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* *Proceedings of the Cambridge Philosophical Society*, vol. v. p. 198 (1884).

Expansions of I and E, §§ 4–6.

§ 4. Since
$$hK = \tfrac{1}{2}(1 - \lambda)\,K, \quad h'K = \tfrac{1}{2}(1 + \lambda)\,K,$$
we may deduce from the expansions of K and G the following expansions of I and E:

$$I = \quad G° \left\{ 1 + \frac{1}{2}\lambda - \frac{3}{2.4}\lambda^2 + \frac{3^2}{2.4.6}\lambda^3 - \frac{3^2.7}{2.4.6.8}\lambda^4 + \frac{3^2.7^2}{2.4.6.8.10}\lambda^5 \right.$$
$$\left. - \frac{3^2.7^2.11}{2.4.6.8.10.12}\lambda^6 + \&c. \right\}$$
$$- \frac{K°}{2} \left\{ 1 - \frac{1}{2}\lambda + \frac{1^2}{2.4}\lambda^2 - \frac{1^2.5}{2.4.6}\lambda^3 + \frac{1^2.5^2}{2.4.6.8}\lambda^4 - \frac{1^2.5^2.9}{2.4.6.8.10}\lambda^5 \right.$$
$$\left. + \frac{1^2.5^2.9^2}{2.4.6.8.10.12}\lambda^6 - \&c. \right\} ;$$

$$E = \quad G° \left\{ 1 - \frac{1}{2}\lambda - \frac{3}{2.4}\lambda^2 - \frac{3^2}{2.4.6}\lambda^3 - \frac{3^2.7}{2.4.6.8}\lambda^4 - \frac{3^2.7^2}{2.4.6.8.10}\lambda^5 \right.$$
$$\left. - \frac{3^2.7^2.11}{2.4.6.8.10.12}\lambda^3 - \&c. \right\}$$
$$+ \frac{K°}{2} \left\{ 1 + \frac{1}{2}\lambda + \frac{1^2}{2.4}\lambda^2 + \frac{1^2.5}{2.4.6}\lambda^3 + \frac{1^2.5^2}{2.4.6.8}\lambda^4 + \frac{1^2.5^2.9}{2.4.6.8.10}\lambda^5 \right.$$
$$\left. + \frac{1^2.5^2.9^2}{2.4.6.8.10.12}\lambda^6 + \&c. \right\} .$$

If we put $k = k'$, then $\lambda = 0$, and these formulæ become
$$I° = G° - \tfrac{1}{2}K°, \quad E° = G° + \tfrac{1}{2}K°,$$
which are the expressions for $I°$ and $E°$ in terms of $K°$ and $G°$.

§ 5. From these expansions we may deduce the following:
$$E + I' = 2G° \left\{ 1 - \frac{1}{2}\lambda - \frac{3}{2.4}\lambda^2 - \frac{3^2}{2.4.6}\lambda^3 - \frac{3^2.7}{2.4.6.8}\lambda^4 \right.$$
$$\left. - \frac{3^2.7^2}{2.4.6.8.10}\lambda^5 - \frac{3^2.7^2.11}{2.4.6.8.10.12}\lambda^6 - \&c. \right\},$$

$$E - I' = K° \left\{ 1 + \frac{1}{2}\lambda + \frac{1^2}{2.4}\lambda^2 + \frac{1^2.5}{2.4.6}\lambda^3 + \frac{1^2.5^2}{2.4.6.8}\lambda^4 \right.$$
$$\left. + \frac{1^2.5^2.9}{2.4.6.8.10}\lambda^5 + \frac{1^2.5^2.9^2}{2.4.6.8.10.12}\lambda^6 + \&c. \right\},$$

with the corresponding expansions for $E' + I$ and $E' - I$ obtained by changing the sign of λ.

§ 6. It may be remarked that we have also

$$I + E = \ 2\,G^\circ \left\{ 1 - \frac{3}{2.4}\,\lambda^2 - \frac{3^2.7}{2.4.6.8}\,\lambda^4 - \frac{3^2.7^2.11}{2.4.6.8.10.12}\,\lambda^6 - \&\mathrm{c.} \right\}$$
$$+ K^\circ \left\{ \frac{1}{2}\,\lambda + \frac{1^2.5}{2.4.6}\,\lambda^3 + \frac{1^2.5^2.9}{2.4.6.8.10}\,\lambda^5 + \&\mathrm{c.} \right\},$$

$$I - E = \ 2\,G^\circ \left\{ \frac{1}{2}\,\lambda + \frac{3^2}{2.4.6}\,\lambda^3 + \frac{3^2.7^2}{2.4.6.8.10}\,\lambda^5 + \&\mathrm{c.} \right\}$$
$$- K^\circ \left\{ 1 + \frac{1^2}{2.4}\,\lambda^2 + \frac{1^2.5^2}{2.4.6.8}\,\lambda^4 + \frac{1^2.5^2.9^2}{2.4.6.8.10.12}\,\lambda^6 + \&\mathrm{c.} \right\},$$

the latter series being the same as that for $-K$ (§ 1), as it ought to be, since $I - E = -K$.

From these two formulæ we may deduce

$$I + E + I' + E' = \ 4\,G^\circ \left\{ 1 - \frac{3}{2.4}\,\lambda^2 - \frac{3^2.7}{2.4.6.8}\,\lambda^4 - \&\mathrm{c.} \right\},$$

$$I + E - I' - E' = \ 2\,K^\circ \left\{ \frac{1}{2}\,\lambda + \frac{1^2.5}{2.4.6}\,\lambda^3 + \frac{1^2.5^2.9}{2.4\ldots10}\,\lambda^5 + \&\mathrm{c.} \right\},$$

$$I - E + I' - E' = -2\,K^\circ \left\{ 1 + \frac{1^2}{2.4}\,\lambda^2 + \frac{1^2.5^2}{2.4.6.8}\,\lambda^4 + \&\mathrm{c.} \right\},$$

$$I - E - I' + E' = \ 4\,G^\circ \left\{ \frac{1}{2}\,\lambda + \frac{3^2}{2.4.6}\,\lambda^3 + \frac{3^2.7^2}{2.4\ldots10}\,\lambda^5 + \&\mathrm{c.} \right\}.$$

The values of the last two series are $-K - K'$ and $-K + K'$, agreeing with § 2. In previous papers I have denoted $\frac{1}{2}(I + E)$ by W (W being a quantity bearing a close analogy to G). The first, third, and fourth series represent therefore W, $W + W'$, and $W - W'$ respectively.

Expansion of $I + G + E$, § 7.

§ 7. For $I + G + E$ we obtain the expansion:

$$I + G + E = \ G^\circ \left\{ 3 - \frac{5}{2.4}\,\lambda^2 - \frac{3^2.13}{2.4.6.8}\,\lambda^4 - \frac{3^2.7^2.21}{2.4.6.8.10.12}\,\lambda^6 \right.$$
$$\left. - \frac{3^2.7^2.11^2.29}{2.4.6\ldots16}\,\lambda^8 - \&\mathrm{c.} \right\}$$
$$+ \frac{K^\circ}{2} \left\{ \frac{1}{2}\,\lambda + \frac{1^2.9}{2.4.6}\,\lambda^3 + \frac{1^2.5^2.17}{2.4.6.8.10}\,\lambda^5 + \frac{1^2.5^2.9^2.25}{2.4.6\ldots14}\,\lambda^7 + \&\mathrm{c.} \right\}.$$