of about two thousand two hundred terms, independently computed to seven decimals by appropriate series, with an increment of e sufficiently small to secure accuracy in interpolation; and by its use, combined with the formulæ

$$F=(1+m)E, \quad \text{or} \quad F=E+\frac{2(E_1-E)}{1-e_1},$$

the first integral F, for any value of e, may be determined without the necessity for an independent tabular development of that function..

ON THE EXPANSION OF $\frac{G}{K}, \frac{K}{G}, \frac{I}{E}$, &c., IN ASCENDING POWERS OF k^2.

By *J. W. L. Glaisher.*

THE object of the present paper is to consider the developments of $\frac{G}{K}, \frac{K}{G}, \frac{I}{E}, \frac{E}{I}$, and of certain other closely connected quantities, in ascending powers of k^2.

Expansion of $\frac{G}{K}$, §§ 1–4.

§ 1. The development of $\frac{E}{K}$ in ascending powers of k was considered by Gudermann in § 94 of his *Theorie der Modular-Functionen und der Modular-Integrale.**

Putting
$$\frac{E}{K}=1-t,$$

Gudermann found that t satisfied the differential equation

$$\frac{dt}{dk}=\frac{(1-2t)k}{k'^2}+\frac{t^2}{kk'^2}$$

Replacing t by $\frac{1}{2}k^2v$, and putting $x=\frac{1}{4}k^2$, he transformed the differential equation into

$$(1-4x)\frac{dv}{dx}+\frac{v-1}{x}-v^2=0.$$

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* p. 180; or *Crelle's Journal*, vol. XVIII., p. 356.

Putting $v = 1 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + \&c.$, substituting in the differential equation, and equating coefficients, Gudermann found

$$a_1 = \frac{1}{2}, \qquad a_5 = \frac{727}{2^5},$$

$$a_2 = 1, \qquad a_6 = \frac{1171}{2^4},$$

$$a_3 = \frac{41}{2^4}, \qquad a_7 = \frac{498409}{2^{11}},$$

$$a_4 = \frac{59}{2^3}, \qquad a_8 = \frac{848479}{2^{10}};$$

whence

$$t = \frac{k^2}{2} + \frac{k^4}{2^4} + \frac{k^6}{2^5} + \frac{41k^8}{2^{11}} + \&c.$$

Gudermann calls attention to the fact that all the denominators are powers of 2.

§ 2. Gudermann's t is equal to $\dfrac{K-E}{K} = -\dfrac{I}{K}$; so that, putting $h = k^2$, his series may be written

$$\frac{I}{K} = -\frac{h}{2} - \frac{h^2}{2^4} - \frac{h^3}{2^5} - \frac{41h^4}{2^{11}} - \&c.$$

Since $G = I + hK$, we find by adding h to each side of the equation,

$$\frac{G}{K} = \frac{h}{2} - \frac{h^2}{2^4} - \frac{h^3}{2^5} - \frac{41h^4}{2^{11}} - \&c.,$$

or, using the coefficients a_1, a_2, ..., whose values are given above,

$$\frac{2G}{K} = h - a_1\frac{h^2}{4} - a_2\frac{h^3}{4^2} - a_3\frac{h^4}{4^3} - a_4\frac{h^5}{4^4} - \&c.$$

§ 3. We may obtain the above formula very simply by the following process, which differs from Gudermann's only in detail.

Let $u = \dfrac{2G}{K}$; then, since

$$\frac{dK}{dh} = \frac{G}{2hh'}, \qquad \frac{dG}{dh} = \frac{K}{2},$$

we have

$$\frac{du}{dh} = \frac{2}{K^2}\left\{K\frac{K}{2} - G\frac{G}{2hh'}\right\}$$

$$= 1 - \frac{1}{hh'}\frac{G^2}{K^2} = 1 - \frac{u^2}{4hh'},$$

so that

$$hh'\left(1 - \frac{du}{dh}\right) = \frac{u^2}{4}.$$

Now, let $u = h - a_1\frac{h^2}{4} - a_2\frac{h^3}{4^2} - a_3\frac{h^4}{4^3} - \&c.;$

then

$$(h - h^2)\left(1 - \frac{du}{dh}\right) = 2a_1\frac{h^2}{4} + (3a_2 - 8a_1)\frac{h^3}{4^2} + (4a_3 - 12a_2)\frac{h^4}{4^3} + \&c.,$$

and

$$\frac{u^2}{4} = \frac{h^2}{4} - 2a_1\frac{h^3}{4^2} - (2a_2 - a_1^2)\frac{h^4}{4^3} - (2a_3 - 2a_2a_1)\frac{h^5}{4^4} - \&c.,$$

whence, by equating coefficients,

$$2a_1 = 1,$$

$$3a_2 = 6a_1,$$

$$4a_3 = 10a_2 + a_1^2,$$

$$5a_4 = 14a_3 + 2a_2a_1,$$

$$6a_5 = 18a_4 + 2a_3a_1 + a_2^2,$$

$$\&c., \qquad \&c.$$

These are the same as Gudermann's equations, from which he obtained the values of a_1, a_2, &c., and on solving them I find $a_1 = \frac{1}{2}$, $a_2 = 1$, $a_3 = \frac{41}{16}$, $a_4 = \frac{59}{8}$, $a_5 = \frac{727}{32}$, agreeing with Gudermann's results.

§ 4. We may, however, also obtain the series for $\frac{2G}{K}$ as the quotient of the two series

$$\frac{2K}{\pi} = 1 + \frac{1^2}{2^2}h + \frac{1^2.3^2}{2^2.4^2}h^2 + \frac{1^2.3^2.5^2}{2^2.4^2.6^2}h^3 + \&c.,$$

$$\frac{2G}{\pi} = \frac{1}{2}h + \frac{1^2}{2^2.4}h^2 + \frac{1^2.3^2}{2^2.4^2.6}h^3 + \&c.$$

For, putting as before

$$\frac{2G}{K} = h - a_1 \frac{h^2}{4} - a_2 \frac{h^3}{4^2} - \&c.,$$

and equating coefficients in the formula

$$\left\{\frac{h}{4} + \frac{1^2}{1^2}\frac{1}{2}\frac{h^2}{4^2} + \frac{1^2.3^2}{1^2.2^2}\frac{1}{3}\frac{h^3}{4^3} + \frac{1^2.3^2.5^2}{1^2.2^2.3^2}\frac{1}{4}\frac{h^4}{4^4} + \&c.\right\}$$

$$= \left\{1 + \frac{h}{4} + \frac{1^2.3^2}{1^2.2^2}\frac{h^2}{4^2} + \frac{1^2.3^2.5^2}{1^2.2^2.3^2}\frac{h^3}{4^3} + \&c.\right\}\left\{\frac{h}{4} - a_1\frac{h^2}{4^2} - a_2\frac{h^3}{4^3} - \&c.\right\},$$

we obtain the following system of equations

$$a_1 = \frac{1}{2},$$

$$a_2 + a_1 = \frac{1^2.3^2}{1^2.2.3},$$

$$a_3 + a_2 + \frac{1^2.3^2}{1^2.2^2}a_1 = \frac{1^2.3^2.5^2}{1^2.2^2.3.4},$$

$$a_4 + a_3 + \frac{1^2.3^2}{1^2.2^2}a_2 + \frac{1^2.3^2.5^2}{1^2.2^2.3^2}a_1 = \frac{1^2.3^2.5^2.7^2}{1^2.2^2.3^2.4.5},$$

$$a_5 + a_4 + \frac{1^2.3^2}{1^2.2^2}a_3 + \frac{1^2.3^2.5^2}{1^2.2^2.3^2}a_2 + \frac{1^2.3^2.5^2.7^2}{1^2.2^2.3^2.4^2}a_1 = \frac{1^2.3^2.5^2.7^2.9^2}{1^2.2^2.3^2.4^2.5.6},$$

&c., &c.

From these equations we find $a_1 = \frac{1}{2}$, $a_2 = 1$, $a_3 = \frac{41}{16}$, &c. as before. They do not give the values of a_1, a_2, ... quite as readily as those employed in the last section; but they afford a reason for the denominators consisting only of powers of 2; for in the above system of equations all the uneven divisors occurring in the denominators are cancelled by the divisors which have already appeared in the numerators, so that only powers of 2 are left in the denominators.*

Expansion of $\log K$, § 5, 6.

§ 5. We may deduce the expansion of $\log K$ from that of $\frac{2G}{K}$. For

$$\frac{2G}{K} = \frac{4hh'}{K}\frac{dK}{dh} = 4hh'\frac{d}{dh}\log K,$$

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* This is true also of the expansion of $\frac{2K}{\pi}$ in ascending powers of h, in which the denominators consist only of powers of 2.

whence, equating coefficients, we obtain the system of equations

$$\begin{aligned}
2b_1 &= 1,\\
3b_2 + b_1^2 &= 4,\\
4b_3 + 2b_2b_1 &= 4^2,\\
5b_4 + 2b_3b_1 + b_2^2 &= 4^3,\\
6b_5 + 2b_4b_1 + 2b_3b_2 &= 4^4,\\
\&c., \qquad & \&c.,
\end{aligned}$$

giving $b_1 = \frac{1}{2}$, $b_2 = \frac{5}{4}$, $b_3 = \frac{59}{16}$, $b_4 = \frac{47}{4}$, &c.

§ 9. By proceeding as in § 4, we have

$$\frac{2K}{G} = \frac{1 + \dfrac{h}{4} + \dfrac{1^2.3^2}{1^2.2^2}\dfrac{h^2}{4^2} + \dfrac{1^2.3^2.5^2}{1^2.2^2.3^2}\dfrac{h^3}{4^3} + \&c.}{\dfrac{h}{4} + \dfrac{1}{2}\dfrac{h^2}{4^2} + \dfrac{1^2.3^2}{1^2.2^2}\dfrac{1}{3}\dfrac{h^3}{4^3} + \dfrac{1^2.3^2.5^2}{1^2.2^2.3^2}\dfrac{1}{4}\dfrac{h^4}{4^4} + \&c.}$$

$$= \frac{4}{h} + b_1 + b_2\frac{h}{4} + b_3\frac{h^2}{4^2} + b_4\frac{h^3}{4^3} + \&c.,$$

whence, by equating coefficients, we obtain the system of equations

$$\begin{aligned}
& b_1 = \frac{1}{2},\\
& b_2 + \frac{b_1}{2} = \frac{1^2.3^2}{1^2.2.3},\\
& b_3 + \frac{b_2}{2} + \frac{1^2.3^2}{1^2.2^2}\frac{b_1}{3} = \frac{1^2.3^2.5^2}{1^2.2^2.3.4},\\
& b_4 + \frac{b_3}{2} + \frac{1^2.3^2}{1^2.2^2}\frac{b_2}{3} + \frac{1^2.3^2.5^2}{1^2.2^2.3^2}\frac{b_1}{4} = \frac{1^2.3^2.5^2.7^2}{1^2.2^2.3^2.4.5},\\
& \qquad \&c., \qquad\qquad \&c.,
\end{aligned}$$

giving the same values of b_1, b_2, b_3, ..., as before. These equations also show that the denominators of these coefficients will consist only of powers of 2.

§ 10. Since

$$\frac{2K}{G} = \frac{4}{G}\frac{dG}{dh} = 4\frac{d}{dh}\log G,$$

we can pass immediately, by integration with respect to h, to the expansion of log G.

We thus find

$$\log G = \log h + b_1 \frac{h}{4} + \frac{b_2}{2}\frac{h^2}{4^2} + \frac{b_3}{3}\frac{h^3}{4^3} + \frac{b_4}{4}\frac{h^4}{4^4} + \&c.,$$

where $b_1, b_2, \ldots$ have the same values as in the last two sections.

§ 11. We have also

$$\frac{K^2}{G^2} = \frac{4}{h^2} + \frac{1}{h} + (4 - b_2)\frac{1}{4} + (4^2 - 2b_3)\frac{h}{4^2} + (4^3 - 3b_4)\frac{h^2}{4^3}$$

$$+ (4^4 - 4b_5)\frac{h^3}{4^4} + \&c.$$

$$= \frac{4}{h^2} + \frac{1}{h} + \frac{11}{2^4} + \frac{69h}{2^7} + \frac{115h^2}{2^8} + \&c.$$

Expansions of $\frac{I}{E}$ *and* $\frac{E}{I}$, §§ 12–15.

§ 12. Let

$$u = -\frac{2I}{E},$$

then, since

$$\frac{dI}{dh} = -\frac{E}{2h'}, \quad \frac{dE}{dh} = \frac{I}{2h},$$

we find at once

$$\frac{du}{dh} = \frac{1}{h'} + \frac{1}{h}\frac{u^2}{4}.$$

Putting

$$u = h + c_1\frac{h^2}{4} + c_2\frac{h^3}{4^2} + c_3\frac{h^4}{4^3} + \&c.$$

and substituting in the differential equation, we find

$$2c_1 = 1 + 4,$$
$$3c_2 = 2c_1 + 4^2,$$
$$4c_3 = 2c_2 + c_1^2 + 4^3,$$
$$5c_4 = 2c_3 + 2c_2c_1 + 4^4,$$
$$\&c., \qquad \&c.,$$

giving $c_1 = \frac{5}{2}$, $c_2 = 7$, $c_3 = \frac{337}{16}$, $c_4 = \frac{533}{8}$, &c.

§ 13. If we derive the expansion from the quotient of the series

$$-\frac{4I}{\pi}=h+\frac{1^2}{1^2}\frac{3}{2}\frac{h^2}{4}+\frac{1^2.3^2}{1^2.2^2}\frac{5}{3}\frac{h^3}{4^2}+\frac{1^2.3^2.5^2}{1^2.2^2.3^2}\frac{7}{4}\frac{h^4}{4^3}+\&c.,$$

$$\frac{2E}{\pi}=1-\frac{h}{4}-\frac{1^2.3}{1^2.2^2}\frac{h^2}{4^2}-\frac{1^2.3^2.5}{1^2.2^2.3^2}\frac{h^3}{4^3}-\frac{1^2.3^2.5^2.7}{1^2.2^2.3^2.4^2}\frac{h^4}{4^4}-\&c.,$$

we find

$$c_1=1+\frac{3}{2},$$

$$c_2=c_1+\frac{1^2.3}{1^2.2^2}\left\{1+\frac{3.5}{3}\right\},$$

$$c_3=c_2+\frac{1^2.3}{1^2.2^2}c_1+\frac{1^2.3^2.5}{1^2.2^2.3^2}\left\{1+\frac{5.7}{4}\right\},$$

$$c_4=c_3+\frac{1^2.3}{1^2.2^2}c_2+\frac{1^2.3^2.5}{1^2.2^2.3^2}c_1=\frac{1^2.3^2.5^2.7}{1^2.2^2.3^2.4^2}\left\{1+\frac{7.9}{5}\right\},$$

&c., &c.,

giving the same values of c_1, c_2, ... as above.

§ 14. Since

$$-\frac{2I}{E}=-\frac{4h}{E}\frac{dE}{dh}=-4h\frac{d}{dh}\log E,$$

we find at once

$$\log E=-\frac{h}{4}-\frac{c_1}{2}\frac{h^2}{4^2}-\frac{c_2}{3}\frac{h^3}{4^3}-\frac{c_3}{4}\frac{h^4}{4^4}-\&c.,$$

and since

$$\frac{u^2}{4}=-\frac{h}{h'}+h\frac{du}{dh},$$

we find

$$\frac{I^2}{E^2}=(2c_1-4)\frac{h^2}{4}+(3c_2-4^2)\frac{h^3}{4^2}+(4c_3-4^3)\frac{h^4}{4^3}$$

$$+(5c_4-4^4)\frac{h^5}{4^4}+\&c.$$

§ 15. Let

$$u=-\frac{2E}{I},$$

then

$$\frac{du}{dh}=-\frac{1}{h}-\frac{u^2}{4h'}.$$

Putting

$$u = \frac{4}{h} + d_1 + d_2\frac{h}{4} + d_3\frac{h^2}{4^2} + d_4\frac{h^3}{4^3} + \&c.,$$

and substituting in the equation

$$(1-h)\frac{du}{dh} = 1 - \frac{1}{h} - \frac{u^2}{4},$$

we find

$$\begin{aligned} 2d_1 &= -5, \\ 3d_2 + d_1^2 &= 4, \\ 4d_3 + 2d_2d_1 &= 4d_2, \\ 5d_4 + 2d_3d_1 + d_2^2 &= 8d_3, \\ 6d_5 + 2d_4d_1 + 2d_3d_2 &= 12d_4, \\ \&c., \qquad & \&c. \end{aligned}$$

Change of h into $-\dfrac{h'}{h}$, § 17.

§ 17. By the change of h into $-\dfrac{h'}{h}$, we convert $\dfrac{G}{K}$ and $\dfrac{I}{E}$ into $\dfrac{I}{h'K}$ and $\dfrac{G}{E}$ respectively. If therefore in the above expansions we replace h by $-\dfrac{h'}{h}$, we obtain the expansions of

$$\frac{I}{h'K}, \quad \frac{h'K}{I}, \quad \frac{G}{E}, \quad \frac{E}{G},$$

in series having their terms alternately positive and negative.

General Remarks, §§ 18, 19.

§ 18. Neither of the methods employed in this paper afford interesting systems of equations for the determination of the coefficients. Perhaps other methods might give simpler and more elegant relations. It is evident that by means of the quotient method we may obtain directly the expansions of the ratios of any two of the quantities K, I, G, E.

§ 19. The investigation of the above developments was suggested by Mr. Childe's paper (pp. 155–164). Mr. Childe puts $F = (1+m)E$, so that his $m = -\frac{I}{E}$, and obtains an expression for m depending upon $\frac{k'}{k}$.

I may mention that Mr. F. W. Newman, in his "Elliptic Integrals" (1889), considers the ratio $\frac{E}{F}$, which he terms the 'ancilla.' He gives a formula (p. 28) which may be written

$$-\frac{I}{kK} = \tfrac{1}{2}k + \tfrac{1}{4}kk_1 + \tfrac{1}{8}kk_1k_2 + \tfrac{1}{16}kk_1k_2k_3 + \&c.,$$

where k_1, k_2, ... are successive moduli formed according to Landen's scale.

It should be added also that the differential equations satisfied by $\frac{G}{K}$, $\frac{K}{G}$, $\frac{I}{E}$, $\frac{E}{I}$ were given by Mr. Kleiber, in vol. XVIII., p. 176, of the *Messenger* (April, 1889), the modular angle θ being the independent variable.

NOTE ON RECIPROCAL LINES.

By Prof. *Cayley*.

If two lines are reciprocal in regard to a quadric surface, then any point on the one line and any point on the other line are harmonics in regard to the surface (viz. the two points and the intersections of their line of junction with the surface form a harmonic range). This is obvious, the polar plane of the first point passes through the second line, and thus the second point is a point in the polar plane of the first point, that is, the two points are harmonics.

But it is worth while to look at the theorem in a different point of view: if the first line meets the surface in the points A, C and the second line meets the surface in the points B, D (in order to the four points being real, the surface must of course be a skew hyperboloid), then AB, BC, CD, DA are lines on the surface, say AB, CD are directrices, and BC, AD are generatrices, the two reciprocal lines being the diagonals AC and BD of the skew quadrilateral. Taking on AC a point P and on BD a point Q, the theorem is that if PQ meets the surface in the points X, Y, then the four points