If the curve consists of two separate portions, one at least must be such as that just found, as otherwise they would divide the plane into regions from each of which an even number of tangents could be drawn. Also from points inside the cusped portion it must be impossible to draw real tangents to the other portion; and this must therefore be an oval convex curve enclosing the former.

Finally, the curve cannot consist of more than two portions, as any additional closed curve added to the previous two would make it possible to draw at least five real tangents from suitably chosen points. This second possible form, consisting of a three-cusped portion surrounded by an oval, will divide the plane into three regions, from which three, one and three real tangents respectively can be drawn.

If the curve is symmetrical, with the same lines of symmetry as an equilateral triangle, its line equation can be written in the form

$$l^3 + m^3 + n^3 - 3klmn = 0.$$

The tangents at the cusps are evidently from symmetry $(0, 1, -1)$, $(1, 0, -1)$, $(1, -1, 0)$ and therefore the coordinates of the cusps are $(k, 1, 1)$, $(1, k, 1)$, $(1, 1, k)$.

When $k = -2$ these lie on the line at infinity and the curve is passing from a form with infinite branches to a closed form. When $k = 0$ the curve is the three-cusped hypocycloid; and when $k = 1$ the three-cusped branch has closed up to a point while the line at infinity appears as part of the curve and gives the first indication of the oval surrounding the three-cusped branch. As k increases from the value unity the three-cusped branch opens out again with the cusps now pointing in the opposite direction to that they did before, while the encircling oval grows smaller, always, however, remaining outside the triangle; the three-cusped branch being entirely inside. As k increases without limits the two branches approach the triangle continually from within and without respectively.

ALGEBRAICAL NOTES.

By *W. Burnside.*

I. *On the Jacobian of two quadratics.*

It is a known theorem that if u and v are quadratic functions of x, then

$$\left(u\frac{dv}{dx} - v\frac{du}{dx}\right)^2 = Au^2 + Buv + Cv^2,$$

where A, B, C are constants. The following process gives a very simple proof of the theorem and determines the constants at the same time.

Put $$u=(x-a)(x-b)=\{x-\tfrac{1}{2}(a+b)\}^2-\tfrac{1}{4}(a-b)^2$$
$$v=(x-c)(x-d)=\{x-\tfrac{1}{2}(c+d)\}^2-\tfrac{1}{4}(c-d)^2,$$

therefore $$\frac{du}{dx}=2x-(a+b)=2\sqrt{\{u+\tfrac{1}{4}(a-b)\}^2}$$

and $$\frac{dv}{dx}=2x-(c+d)=2\sqrt{\{v+\tfrac{1}{4}(c-d)^2\}};$$

therefore

$$\tfrac{1}{2}\left(u\frac{dv}{dx}-v\frac{du}{dx}\right)=u\sqrt{\{v+\tfrac{1}{4}(c-d)\}^2}-v\sqrt{\{u+\tfrac{1}{4}(a-b)\}^2}$$

and $\tfrac{1}{2}(a+b-c-d)=\sqrt{\{v+\tfrac{1}{4}(c-d)\}^2}-\sqrt{\{u+\tfrac{1}{4}(a-b)^2\}}$;

therefore

$$\left(u\frac{dv}{dx}-v\frac{du}{dx}\right)^2-uv(a+b-c-d)^2=u^2(c-d)^2+v^2(a-b)^2$$
$$-uv\{(c-d)^2+(a-b)^2\},$$

or

$$A=(c-d)^2,\ B=(a+b-c-d)^2-(a-b)^2-(c-d)^2,\ C=(a-b)^2$$

II. *On a system of simultaneous equations.*

In the recent Mathematical Tripos I proposed for solution the simultaneous equations

$$\frac{ax+by+cz}{x}=\frac{bx+cy+az}{y}=\frac{cx+ay+bz}{z}.$$

The form of solution of these, which clearly depends only on a quadratic equation, suggests the solution of the more general equations

$$\frac{a_1x_1+a_2x_2+\ldots+a_nx_n}{x_1}=\frac{a_2x_1+\ldots+a_1x_n}{x_2}=\ldots$$
$$\ldots=\frac{a_nx_1+a_1x_2+\ldots+a_{n-1}x_n}{x_n}.$$

Thus, as guided by analogy, try

$$\frac{x_1}{A+B}=\frac{x_2}{A\omega+B\omega^{n-1}}=\frac{x_3}{A\omega^2+B\omega^{n-2}}=\ldots,$$

where $\omega^n=1$.

Then

$$\frac{a_1x_1+\ldots+a_nx_n}{x_1}=\frac{A(a_1+a_2\omega+a\omega^2+\ldots)+B(a_1+a_2\omega^{n-1}+a_2\omega^{n-2}+\ldots)}{A+B},$$

$$\frac{a_2x_1+\ldots+a_1x_n}{x_2}=\frac{A(a_2+a_3\omega+a_4\omega^2+\ldots)+B(a_2+a_3\omega^{n-1}+a_4\omega^{n-2}+\ldots)}{A\omega+B\omega^{n-1}},$$

$$=\frac{A\omega^{n-1}(a_1+a_2\omega+\ldots)+B\omega(a_1+a_2\omega^{n-1}+\ldots)}{A\omega+B\omega^{n-1}},$$

....

Hence, obviously, if

$$A^2=a_1+a_2\omega^{n-1}+a_6\omega^{n-2}+\ldots$$

$$B^2=a_1+a_2\omega+a_3\omega^2+\ldots$$

these fractions are all equal to the same quantity, namely, AB; and with the values so obtained of A and B the trial solution satisfies the equations. The expressions A^2 and B^2 are conjugate imaginaries, so that the ratios of the unknown are real for either determination of the square roots A and B. The complete system of n solutions will be as follows:—

(i) n even,

$$x_1=x_2=x_3=\ldots$$

$$x_1=-x_2=x_3=-x_4=\ldots$$

$$\frac{x_1}{\sqrt{(a_1+a_2\omega+\ldots)}\pm\sqrt{(a_1+a_2\omega^{n-1}+\ldots)}}$$

$$=\frac{x_2}{\omega^{n-1}\sqrt{(a_1+a_2\omega+\ldots)}\pm\omega\sqrt{(a_1+a_2\omega^{n-1}+\ldots)}}=\ldots,$$

where $\omega=e^{\frac{2ir\pi}{n}}$ $(r=1, 2, \ldots, \tfrac{1}{2}n-1)$.

(ii) n odd, the same expressions omitting the second line; while r takes the values $1, 2, \ldots, \tfrac{1}{2}(n-1)$. The square roots are all to be taken with their real parts positive.