RECURRING RELATIONS INVOLVING SUMS OF POWERS OF DIVISORS.

(Third Paper).

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Introduction, § 1.

§ 1. Two papers which were published under the above title in the last volume of the *Messenger* (pp. 129, 177) contain recurring relations connecting together the functions

$$\sigma_m\,(n),\ \sigma_m\,(n-1),\ \sigma_m\,(n-3),\ \sigma_m\,(n-6),\ \ldots$$
$$\sigma_{m-2}(n),\ \sigma_{m-2}(n-1),\ \sigma_{m-2}(n-3),\ \sigma_{m-2}(n-6),\ \ldots$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,$$

where n is any number, m is any uneven number, $\sigma_m(n)$ denotes the sum of the m^{th} powers of the divisors of n, and the numbers 1, 3, 6, ..., which occur in the arguments, are the triangular numbers.

The present paper contains results of a similar character to those given in the second of these papers, the principal difference being that the functions involved are not only of the form $\sigma_m(n)$, but also of the form $n^r\sigma_m(n)$.

Formulæ involving $\sigma(n)$, $n\sigma(n)$, $n^2\sigma(n)$, *&c.* §§ 2–4.

§ 2. The group of theorems that I have obtained may be written

I.

$$\sigma(n) - 3\sigma(n-1) + 5\sigma(n-3) - 7\sigma(n-6) + \&c.$$
$$= \left[(-1)^{\frac{1}{2}(f+1)}\frac{f^3-f}{24}\right].$$

II.

$$\sigma_3(n) - 3\sigma_3(n-1) + 5\sigma_3(n-3) - 7\sigma_3(n-6) + \&c.$$
$$-2\,\{n\sigma(n) - 3(n-1)\,\sigma(n-1) + 5(n-3)\,\sigma(n-3) - \&c.\}$$
$$= \left[(-1)^{\frac{1}{2}(f-1)}\frac{f^5-f}{240}\right].$$

III.

$$\begin{aligned}&\sigma_5(n) - 3\sigma_5(n-1) + 5\sigma_5(n-3) - 7\sigma_5(n-6) + \&c.\\ &- 10\{n\sigma_3(n) - 3(n-1)\sigma_3(n-1) + 5(n-3)\sigma_3(n-3) - \&c.\}\\ &+ \tfrac{40}{3}\{n^2\sigma(n) - 3(n-1)^2\sigma(n-1) + 5(n-3)^2\sigma(n-3) - \&c.\}\\ &= \left[(-1)^{\frac{1}{2}(f+1)}\frac{f^7-f}{504}\right].\end{aligned}$$

IV.

$$\begin{aligned}&\sigma_7(n) - 3\sigma_7(n-1) + 5\sigma_7(n-3) - 7\sigma_7(n-6) + \&c.\\ &- \tfrac{126}{5}\{n\sigma_5(n) - 3(n-1)\sigma_5(n-1) + 5(n-3)\sigma_5(n-3) - \&c.\}\\ &+ \tfrac{756}{5}\{n^2\sigma_3(n) - 3(n-1)^2\sigma_3(n-1) + 5(n-3)^2\sigma_3(n-3) - \&c.\}\\ &- 168\{n^3\sigma(n) - 3(n-1)^3\sigma(n-1) + 5(n-3)^3\sigma(n-3) - \&c.\}\\ &= \left[(-1)^{\frac{1}{2}(f-1)}\frac{f^9-f}{480}\right].\end{aligned}$$

V.

$$\begin{aligned}&\sigma_9(n) - 3\sigma_9(n-1) + 5\sigma_9(n-3) - 7\sigma_9(n-6) + \&c.\\ &- 50\{n\sigma_7(n) - 3(n-1)\sigma_7(n-1) + 5(n-3)\sigma_7(n-3) - \&c.\}\\ &+ 720\{n^2\sigma_5(n) - 3(n-1)^2\sigma_5(n-1) + 5(n-3)^2\sigma_5(n-3) - \&c.\}\\ &- 3360\{n^3\sigma_3(n) - 3(n-1)^3\sigma_3(n-1) + 5(n-3)^3\sigma_3(n-3) - \&c.\}\\ &+ 3360\{n^4\sigma(n) - 3(n-1)^4\sigma(n-1) + 5(n-3)^4\sigma(n-3) - \&c.\}\\ &= \left[(-1)^{\frac{1}{2}(f+1)}\frac{f^{11}-f}{264}\right].\end{aligned}$$

When n is not a triangular number, the right-hand member in these equations, which is inclosed in [ ], is equal to zero; but when n is a triangular number, $\frac{1}{2}g(g+1)$, the right-hand member is to be included, f being $= 2g+1$. Thus f may be defined as the coefficient of $\sigma_m(0)$ in the first series, when this term occurs*.

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* When n is a triangular number, $f^2 = 8n+1$; we may therefore dispense with the additional term entirely in these formulæ (the right-hand member of the equations being then zero in all cases) if we assign to $\sigma(0)$, $\sigma_3(0)$, &c., the values:

$$\begin{aligned}\sigma(0) &= \tfrac{1}{3}n,\\ \sigma_3(0) &= -\frac{(8n+1)^2-1}{240},\\ \sigma_5(0) &= \frac{(8n+1)^3-1}{504},\\ \sigma_7(0) &= -\frac{(8n+1)^4-1}{480},\\ \sigma_9(0) &= \frac{(8n+1)^5-1}{264}.\end{aligned}$$

Only one such term can occur in each formula, as all the terms of zero argument, except the one which has the highest suffix, have zero as coefficient.

§ 3. As numerical examples of the formulæ, let $n=5$ and 6, so that $f=7$, in the latter case; we thus find:

I.

$$\sigma(5)-3\sigma(4)+5\sigma(2)=0;$$

$$\sigma(6)-3\sigma(5)+5\sigma(3)=\tfrac{1}{24}(7^3-7).$$

II.

$$\begin{aligned}&\sigma_3(5)-3\sigma_3(4)\quad+5\sigma_3(2)\\&-2\{5\sigma(5)-3.4\sigma(4)+5.2\sigma(2)\}=0;\end{aligned}$$

$$\begin{aligned}&\sigma_3(6)-3\sigma_3(5)\quad+5\sigma_3(3)\\&-2\{6\sigma(6)-3.5\sigma(5)+5.3\sigma(3)\}=-\tfrac{1}{240}(7^5-7).\end{aligned}$$

III.

$$\begin{aligned}&\sigma_5(5)-3\sigma_5(4)\quad+5\sigma_5(2)\\&-10\{5\sigma_3(5)-3.4\sigma_3(4)+5.2\sigma_3(2)\}\\&+\tfrac{40}{3}\{5^2\sigma(5)-3.4^2\sigma(4)+5.2^2\sigma(2)\}=0;\end{aligned}$$

$$\begin{aligned}&\sigma_5(6)-3\sigma_5(5)\quad+5\sigma_5(3)\\&-10\{6\sigma_3(6)-3.5\sigma_3(5)+5.3\sigma_3(3)\}\\&+\tfrac{40}{3}\{6^2\sigma(6)-3.5^2\sigma(5)+5.3^2\sigma(3)\}=\tfrac{1}{504}(7^7-7).\end{aligned}$$

IV.

$$\begin{aligned}&\sigma_7(5)-3\sigma_7(4)\quad+5\sigma_7(2)\\&-\tfrac{126}{5}\{5\,\sigma_5(5)-3.4\,\sigma_5(4)+5.2\sigma_5(2)\}\\&+\tfrac{756}{5}\{5^2\sigma_3(5)-3.4^2\sigma_3(4)+5.2^2\sigma_3(2)\}\\&-168\{5^3\sigma(5)-3.4^3\sigma(4)+5.2^3\sigma(2)\}=0;\end{aligned}$$

$$\begin{aligned}&\sigma_7(6)-3\sigma_7(5)\quad+5\sigma_7(3)\\&-\tfrac{126}{5}\{6\,\sigma_5(6)-3.5\,\sigma_5(5)+5.3\,\sigma_5(3)\}\\&+\tfrac{756}{5}\{6^2\sigma_3(6)-3.5^2\sigma_3(5)+5.3^2\sigma_3(3)\}\\&-168\{6^3\sigma(6)-3.5^3\sigma(5)+5.3^3\sigma(3)\}=-\tfrac{1}{480}(7^9-7).\end{aligned}$$

V.

$$
\begin{aligned}
&\sigma_9(5) - 3\sigma_9(4) \qquad + 5\sigma_9(2)\\
-\;\;& 50\,\{5\,\sigma_7(5) - 3.4\,\sigma_7(4) + 5.2\,\sigma_7(2)\}\\
+\;& 720\,\{5^2\sigma_5(5) - 3.4^2\sigma_5(4) + 5.2^2\sigma_2(2)\}\\
-\;& 3360\,\{5^3\sigma_3(5) - 3.4^3\sigma_3(4) + 5.2^3\sigma_3(2)\}\\
+\;& 3360\,\{5^4\sigma\;(5) - 3.4^4\sigma\;(4) + 5.2^4\sigma\;(2)\} = 0\,;
\end{aligned}
$$

$$
\begin{aligned}
&\sigma_9(6) - 3\sigma_9(5) \qquad + 5\sigma_9(3)\\
-\;\;& 50\,\{6\,\sigma_7(6) - 3.5\,\sigma_7(5) + 5.3\,\sigma_7(3)\}\\
+\;& 720\,\{6^2\sigma_5(6) - 3.5^2\sigma_5(5) + 5.3^2\sigma_5(3)\}\\
-\;& 3360\,\{6^3\sigma_3(6) - 3.5^3\sigma_3(5) + 5.3^3\sigma_3(3)\}\\
+\;& 3360\,\{6^4\sigma\;(6) - 3.5^4\sigma\;(5) + 5.3^4\sigma\;(3)\} = \tfrac{1}{264}\,(7^{11} - 7).
\end{aligned}
$$

§ 4. In the above formulæ all the series are of the form

$$\phi\,(n) - 3\phi\,(n-1) + 5\phi\,(n-3) - 7\phi\,(n-6) + \&c.,$$

but the functions occurring in the successive series are of the forms $\sigma_m(n)$, $n\sigma_{m-2}(n)$, $n^2\sigma_{m-4}(n)$, &c.; in the formulæ given in the second of the papers, referred to in § 1, the functions are of the simpler form $\sigma_m(n)$, $\sigma_{m-2}(n)$, $\sigma_{m-4}(n)$, ..., but the coefficients in the successive series are 1, 3, 5, ... ; 1^3, 3^3, 5^3, ... ; 1^5, 3^5, 5^5, ..., &c.

Corresponding formulæ involving σ (n) only, §§ 5, 6.

§ 5. For the sake of comparison I now give the corresponding formulæ derived from the general formulæ on p. 181 for the cases $m = 1, 3, 5, 7, 9$.

I.

$$
\begin{aligned}
&\sigma\,(n) - 3\sigma\,(n-1) + 5\sigma\,(n-3) - 7\sigma\,(n-6) + \&c.\\
&\qquad = \left[(-1)^{\frac{1}{2}(f+1)}\frac{f^3 - f}{24}\right].
\end{aligned}
$$

II.

$$
\begin{aligned}
&\sigma_3(n) - 3\sigma_3(n-1) + 5\sigma_3(n-3) - 7\sigma_3(n-6) + \&c.\\
&+ \tfrac{1}{4}\,\{\sigma\;(n) - 3^3\sigma(n-1) + 5^3\sigma(n-3) - 7^3\sigma(n-6) + \&c.\}\\
&\qquad = \left[(-1)^{\frac{1}{2}(f+1)}\frac{3f^5 - 5f^3 + 2f}{480}\right].
\end{aligned}
$$

III.

$$\begin{aligned}&\sigma_5(n) - 3\,\sigma_5(n-1) + 5\,\sigma_5(n-3) - 7\,\sigma_5(n-6) + \&c.\\ &+ \tfrac{5}{6}\,\{\sigma_3(n) - 3^3\sigma_3(n-1) + 5^3\sigma_3(n-3) - 7^3\sigma_3(n-6) + \&c.\}\\ &+ \tfrac{1}{16}\,\{\sigma\,(n) - 3^5\sigma\,(n-1) + 5^5\sigma\,(n-3) - 7^5\sigma\,(n-6) + \&c.\}\\ &= \left[(-1)^{\frac{1}{2}(f+1)}\frac{9f^7 - 21f^5 + 28f^3 - 16f}{8064}\right].\end{aligned}$$

IV.

$$\begin{aligned}&\sigma_7(n) - 3\sigma_7(n-1) + 5\sigma_7(n-3) - 7\,\sigma_7(n-6) + \&c.\\ &+ \tfrac{7}{4}\,\{\sigma_5(n) - 3^3\sigma_5(n-1) + 5^3\sigma_5(n-3) - 7^3\sigma_5(n-6) + \&c.\}\\ &+ \tfrac{7}{16}\,\{\sigma_3(n) - 3^5\sigma_3(n-1) + 5^5\sigma_3(n-3) - 7^5\sigma_3(n-6) + \&c.\}\\ &+ \tfrac{1}{64}\,\{\sigma\,(n) - 3^7\sigma\,(n-1) + 5^7\sigma\,(n-3) - 7^7\sigma\,(n-6) + \&c.\}\\ &= \left[(-1)^{\frac{1}{2}(f+1)}\frac{35f^9 - 105f^7 + 294f^5 - 560f^3 + 336f}{161280}\right].\end{aligned}$$

V.

$$\begin{aligned}&\sigma_9(n) - 3\,\sigma_9(n-1) + 5\,\sigma_9(n-3) - 7\,\sigma_9(n-6) + \&c.\\ &+ 3\,\{\sigma_7(n) - 3^3\sigma_7(n-1) + 5^3\sigma_7(n-3) - 7^3\sigma_7(n-6) + \&c.\}\\ &+ \tfrac{63}{40}\,\{\sigma_5(n) - 3^5\sigma_5(n-1) + 5^5\sigma_5(n-3) - 7^5\sigma_5(n-6) + \&c.\}\\ &+ \tfrac{3}{16}\,\{\sigma_3(n) - 3^7\sigma_3(n-1) + 5^7\sigma_3(n-3) - 7^7\sigma_3(n-6) + \&c.\}\\ &+ \tfrac{1}{256}\,\{\sigma\,(n) - 3^9\sigma\,(n-1) + 5^9\sigma\,(n-3) - 7^9\sigma\,(n-6) + \&c.\}\\ &= \left[(-1)^{\frac{1}{2}(f+1)}\frac{15f^{11} - 55f^9 + 264f^7 - 1056f^5 + 2112f^3 - 1280f}{337920}\right].\end{aligned}$$

§ 6. These formulæ are particular cases of the general theorem given on p. 181 of Vol. XX; but I have not obtained the general theorem to which the formulæ in § 2 belong. The latter results were obtained by a somewhat complicated method which does not seem likely to lead readily to the general formula. It will be noticed that in the formulæ of § 2 the additional term, on the right-hand side, is especially simple, the expression involving f being of the form $f^p - f$.

The first equation is common to both systems of results, and also to the system given in the first paper (*Messenger*, Vol. XX., pp. 129).

Before considering further the relations between the systems in §§ 2 and 5, it is convenient to give an account of a notation which I have found useful for the representation of the series.

III.

$$J\sigma_5(n) - 10J\{n\sigma_3(n)\} + \tfrac{40}{3}J\{n^2\sigma(n)\} = \left[(-1)^{\frac{1}{2}(f+1)}\frac{f^7-f}{504}\right].$$

IV.

$$J\sigma_7(n) - \tfrac{126}{5}J\{n\sigma_5(n)\} + \tfrac{756}{5}J\{n^2\sigma_3(n)\}$$
$$- 168J\{n^3\sigma(n)\} = \left[(-1)^{\frac{1}{2}(f-1)}\frac{f^9-f}{480}\right].$$

V.

$$J\sigma_9(n) - 50J\{n\sigma_7(n)\} + 720J\{n^2\sigma_5(n)\} - 3360J\{n^3\sigma_3(n)\}$$
$$+ 3360J\{n^4\sigma(n)\} = \left[(-1)^{\frac{1}{2}(f+1)}\frac{f^{11}-f}{264}\right],$$

In the same notation the formulæ of § 5 are

I.

$$J\sigma(n) = \left[(-1)^{\frac{1}{2}(f+1)}\frac{f^3-f}{24}\right].$$

II.

$$J\sigma_3(n) + \tfrac{1}{4}J_3\sigma(n) = \left[(-1)^{\frac{1}{2}(f+1)}\frac{3f^5-5f^3+2f}{480}\right].$$

III.

$$J\sigma_5(n) + \tfrac{5}{6}J_3\sigma_3(n) + \tfrac{1}{16}J_5\sigma(n)$$
$$= \left[(-1)^{\frac{1}{2}(f+1)}\frac{9f^7-21f^5+28f^3-16f}{8064}\right].$$

IV.

$$J\sigma_7(n) + \tfrac{7}{4}J_3\sigma_5(n) + \tfrac{7}{16}J_5\sigma_3(n) + \tfrac{1}{64}J_7\sigma(n)$$
$$= \left[(-1)^{\frac{1}{2}(f+1)}\frac{35f^9-105f^7+294f^5-560f^3+336f}{161280}\right].$$

V.

$$J\sigma_9(n) + 3J_3\sigma_7(n) + \tfrac{63}{40}J_5\sigma_5(n) + \tfrac{3}{16}J_7\sigma_3(n) + \tfrac{1}{256}J_9\sigma(n)$$
$$= \left[(-1)^{\frac{1}{2}(f+1)}\frac{15f^{11}-55f^9+264f^7-1056f^5+2112f^3-1280f}{337920}\right].$$

Values of $Jn\phi(n)$, $Jn^2\phi(n)$, &c. in terms of $J\phi(n)$, $J'\phi(n)$, &c., § 12.

§ 12. We have

$$J\{n\phi(n)\} = n\phi(n) - 3(n-1)\phi(n-1) + 5(n-3)\phi(n-3) - \&c.$$
$$= n\{\phi(n) - 3\phi(n-1) + 5\phi(n-3) - \&c.\}$$
$$+ 3\phi(n-1) - 5.3\phi(n-3) + 7.6\phi(n-6) - \&c.$$
$$= nJ\phi(n) + J'\phi(n).$$

Similarly, we find

$$J\{n^2\phi(n)\} = n^2J\phi(n) + 2nJ'\phi(n) + J''\phi(n),$$
$$J\{n^3\phi(n)\} = n^3J\phi(n) + 3n^2J'\phi(n) + 3nJ''\phi(n) + J'''\phi(n),$$

and, in general,

$$J\{n^r\phi(n)\} = \{n^rJ + rn^{r-1}J' + \frac{r(r-1)}{2!}n^{r-2}J'' + \ldots + J^{(r)}\}\phi(n).$$

By means of the formulæ in § 10, we can pass from J', J'', &c. to J_3, J_5, &c., and we can thus express the formulæ in § 2 by means of series of the form $J_m\phi(n)$.

Relations between the two systems of recurring formulæ, §§13–18.

§ 13. Taking the second formulæ of § 2, viz.,

$$J\sigma_3(n) - 2J\{n\sigma(n)\} = (-1)^{\frac{1}{2}(f-1)}\frac{f^5 - f}{240},$$

we have

$$J\sigma_3(n) - 2nJ\sigma(n) - 2J'\sigma(n) = \left[(-1)^{\frac{1}{2}(f-1)}\frac{f^5 - f}{240}\right].$$

Now $$J\sigma(n) = \left[(-1)^{\frac{1}{2}(f+1)}\frac{f^3 - f}{24}\right];$$

and, when n is a triangular number, $n = \dfrac{f^2 - 1}{8}$.

Thus,

$$J\sigma_3(n) - 2J'\sigma(n) = \left[(-1)^{\frac{1}{2}(f-1)}\left\{\frac{f^5 - f}{240} - 2\frac{(f^3 - f)(f^2 - 1)}{24 \times 8}\right\}\right]$$
$$= \left[(-1)^{\frac{1}{2}(f+1)}\frac{3f^5 - 10f^3 + 7f}{480.}\right].$$

Now, by § 10,

$$J'\phi(n) = \tfrac{1}{8}(J_1 - J_3)\,\phi(n),$$

so that the above equation becomes

$$J\sigma_3(n) + \tfrac{1}{4}J_3\sigma(n) = \left[(-1)^{\frac{1}{2}(f+1)}\frac{3f^5 - 10f^3 + 7f}{480}\right] + \tfrac{1}{4}J\sigma(n)$$

$$= \left[(-1)^{\frac{1}{2}(f+1)}\frac{3f^5 - 5f^3 + 2f}{480}\right],$$

which is the same as the second equation of § 5.

§ 14. The third formula however in § 2 does not appear to be derivable from the formulæ in § 5. For, omitting the additional term which occurs only when n is a triangular number, *i.e.* assuming that n is not a triangular number, the third formula of § 2 is

$$J\sigma_5(n) - 10J\{n\sigma_3(n)\} + \tfrac{40}{3}J\{n^2\sigma(n)\} = 0,$$

that is,

$$J\sigma_5(n) - 10\{nJ\sigma_3(n) + J'\sigma_3(n)\}$$
$$+ \tfrac{40}{3}\{n^2J\sigma(n) + 2nJ'\sigma(n) + J''\sigma(n)\} = 0.$$

But $J\sigma(n) = 0$, so that this equation is

$$J\sigma_5(n) - 10nJ\sigma_3(n) - 10J'\sigma_3(n) + \tfrac{80}{3}nJ'\sigma(n) + \tfrac{40}{3}J''\sigma(n) = 0.$$

§ 15. Now the second and third formulæ of § 5 when expressed by means of J', J'' are:

$$J\sigma_3(n) - 2J'\sigma(n) = 0,$$

$$J\sigma_5(n) - \tfrac{20}{3}J'\sigma_3(n) + 4J''\sigma(n) + \tfrac{2}{3}J'\sigma(n) = 0,$$

it being supposed as before that n is not a triangular number.

Subtracting the last formula in the preceding section from the second of these formulæ, we find

$$\tfrac{10}{3}J'\sigma_3(n) + 10nJ\sigma_3(n) - \tfrac{28}{3}J''\sigma(n) + \tfrac{2}{3}J'\sigma(n) - \tfrac{80}{3}nJ'\sigma(n) = 0.$$

Replacing $J\sigma_3(n)$ by $2J'\sigma(n)$, this becomes

$$\tfrac{10}{3}J'\sigma_3(n) - \tfrac{20}{3}nJ'\sigma(n) + \tfrac{2}{3}J'\sigma(n) - \tfrac{28}{3}J''\sigma(n) = 0,$$

that is,

$$J'\sigma_3(n) = 2nJ'\sigma(n) - \tfrac{1}{5}J'\sigma(n) + \tfrac{14}{5}J''\sigma(n).$$

The presence of the factor n in the first term on the right-hand side of this equation shows that it cannot be obtained by combining the formulæ of § 5.

§16. Writing the series in the above equation at full length, and dividing throughout by 3, it becomes

$$\sigma_3(n-1) - 5\sigma_3(n-3) + 14\sigma_3(n-6) - \&c.$$
$$= (2n - \tfrac{1}{5})\{\sigma(n-1) - 5\sigma(n-3) + 14\sigma(n-6) - \&c.\}$$
$$- \tfrac{14}{5}\{\sigma(n-1) - 15\sigma(n-3) + 84\sigma(n-6) - \&c.\} = 0,$$

which holds good for all values of n which are not triangular numbers.

§17. This formula may be conveniently expressed by means of J_1 and J_3. Replacing J' and J'' by $\frac{1}{8}(J_1 - J_3)$, and $\frac{1}{64}(J_1 - 2J_3 + J_5)$, and putting $J_1\sigma(n) = 0$, it becomes

$$J_1\sigma_3(n) - J_3\sigma_3(n) = -2nJ_3\sigma(n) - \tfrac{1}{2}J_3\sigma(n) + \tfrac{7}{20}J_5\sigma(n).$$

But (by § 5), $\quad J_1\sigma_3(n) + \frac{1}{4}J_3\sigma(n) = 0,$

whence we find

$$J_3\sigma_3(n) = (2n + \tfrac{1}{4})J_3\sigma(n) - \tfrac{7}{20}J_5\sigma(n),$$

that is, writing the series at full length,

$$\sigma_3(n) - 3^3\sigma_3(n-1) + 5^3\sigma_3(n-3) - 7^3\sigma_3(n-6) + \&c.$$
$$= (2n + \tfrac{1}{4})\{\sigma(n) - 3^3\sigma(n-1) + 5^3\sigma(n-3) - \&c.\}$$
$$- \tfrac{7}{20}\{\sigma(n) - 3^5\sigma(n-1) + 5^5\sigma(n-3) + \&c.\},$$

which holds good for all values of n which are not triangular numbers.

§18. For example, putting $n = 4$, this formula gives

$$\sigma_3(4) - 3^3\sigma_3(3) + 5^3\sigma_3(1) = \tfrac{33}{4}\{\sigma(4) - 3^3\sigma(3) + 5^3\sigma(1)\}$$
$$- \tfrac{7}{20}\{\sigma(4) - 3^5\sigma(3) + 5^5\sigma(1)\},$$

which is easily verified.

The function G, § 19.

§19. Corresponding to the formulæ in § 2, we have also recurring formulæ in which the arguments of all the numbers from 1 to n are involved. These theorems can be conveniently expressed by means of the function G, defined by the equation

$$G\phi(n) = \phi(n) - 2\phi(n-1) - 2\phi(n-2) + 3\phi(n-3)$$
$$+ 3\phi(n-4) + 3\phi(n-5) - 4\phi(n-6) - \ldots - 4\phi(n-9)$$
$$+ 5\phi(n-10) + \ldots + (-1)^{r-1}r\phi(1),$$

in which the coefficient of the first term is unity, that of the next two is 2, of the next three 3, of the next four 4, and so on, the groups of 1, 2, 3, 4, ... terms alternating in sign.

Recurring formulæ involving the divisors of all the numbers from unity to n, §§ 20–22.

§ 20. Using the above notation and denoting by s what would be the coefficient of the term $\phi(0)$ if the series were continued one term further (so that, if r be the coefficient of $\phi(1)$, then $s = r$ unless the term $\pm r\phi(1)$ is the last of its group in which case $s = r + 1$), we have the following formulæ:

I.

$$G\sigma(n) = \frac{(-1)^{s-1}s - 1^3 + 3^3 - 5^3 + \ldots + (-1)^s(2s-1)^3}{24}.$$

II.

$$G\sigma_3(n) - 2G\{n\sigma(n)\} = -\frac{(-1)^{s-1}s - 1^5 + 3^5 - 5^5 + \ldots + (-1)^s(2s-1)^5}{240}.$$

III.

$$G\sigma_5(n) - 10G\{n\sigma_3(n)\} + \tfrac{40}{3}G\{n^2\sigma(n)\} = \frac{(-1)^{s-1}s - 1^7 + 3^7 - 5^7 + \ldots + (-1)^s(2s-1)^7}{504}.$$

IV.

$$G\sigma_7(n) - \tfrac{126}{5}G\{n\sigma_5(n)\} + \tfrac{756}{5}G\{n^2\sigma_3(n)\} - 168G\{n^3\sigma(n)\} = -\frac{(-1)^{s-1}s - 1^9 + 3^9 - 5^9 + \ldots + (-1)^s(2s-1)^9}{480}.$$

V.

$$G\sigma_9(n) - 50G\{n\sigma_7(n)\} + 720G\{n^2\sigma_5(n)\} - 3360G\{n^3\sigma_3(n)\} + 3360G\{n^4\sigma(n)\} = \frac{(-1)^{s-1}s - 1^{11} + 3^{11} - 5^{11} + \ldots + (-1)^s(2s-1)^{11}}{264}.*$$

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* It is evident that if we put

$$\sigma(0) = -\tfrac{1}{24},\quad \sigma_3(0) = \tfrac{1}{240},\quad \sigma_5(0) = -\tfrac{1}{504},$$
$$\sigma_7(0) = \tfrac{1}{480},\quad \sigma_9(0) = -\tfrac{1}{264},$$

we may omit the term $(-1)^{s-1}s$ from each equation, the right-hand members thus becoming simply

$$\frac{-1^3 + 3^3 - 5^3 + \ldots + (-1)^s(2s-1)^3}{24},$$

$$\frac{1^5 - 3^5 + 5^5 - \ldots + (-1)^{s-1}(2s-1)^5}{240},$$

&c. &c.

§ 21. Writing the series in the first two formulæ at full length, as specimens, we have

I.

$$\sigma(n) - 2\sigma(n-1) - 2\sigma(n-2) + 3\sigma(n-3) + 3\sigma(n-4)$$
$$+ 3\sigma(n-5) - 4\sigma(n-6) - 4\sigma(n-7) - \&c.$$
$$= \frac{(-1)^{s-1}s - 1^3 + 3^3 - 5^3 + \ldots + (-1)^s(2s-1)^3}{24}.$$

II.

$$\sigma_3(n) - 2\sigma_3(n-1) - 2\sigma_3(n-2) + 3\sigma_3(n-3) + 3\sigma_3(n-4) + \&c.$$
$$-2\{n\sigma(n) - 2(n-1)\sigma(n-1) - 2(n-2)\sigma(n-2)$$
$$+ 3(n-3)\sigma(n-3) + 3(n-4)\sigma(n-4) + \&c.\}$$
$$= -\frac{(-1)^{s-1}s - 1^5 + 3^5 - 5^5 + \ldots + (-1)^s(2s-1)^5}{240}.$$

As a numerical example, putting $n=3$ in the third formula, we find

$$\sigma_5(3) - 2\sigma_5(2) - 2\sigma_5(1)$$
$$-10\{3\sigma_3(3) - 2.2\sigma_3(2) - 2\sigma_3(1)\}$$
$$+ \tfrac{40}{3}\{3^2\sigma(3) - 2.2^2\sigma(2) - 2\sigma(1)\} = \frac{3 - 1^7 + 3^7 - 5^7}{504},$$

which is easily verified, each side being equal to $-\frac{452}{3}$.

§ 22. The first formula was given in Vol. V., p. 113, of the *Proc. Camb. Phil. Soc.* (1884), the right-hand member of the equation being there expressed in the form

$$(-1)^s \tfrac{1}{6}(s^3 - s),$$

which may be readily identified with the form in § 20.

Analytical formulæ, §§ 23, 24.

§ 23. The analytical results from which the recurring formulæ in §§ 2 and 20 were derived are:

I.

$$\frac{1 - 3^3q + 5^3q^2 - 7^3q^3 + \&c.}{1 - 3q + 5q^2 - 7q^3 + \&c.} = 1 - 24\Sigma\sigma(n)q^n.$$

II.

$$\frac{1 - 3^5q + 5^5q^2 - 7^5q^3 + \&c.}{1 - 3q + 5q^2 - 7q^3 + \&c.} = 1 + 240\Sigma\sigma_3(n)q^n - 480\Sigma n\sigma(n)q^n.$$

III.

$$\frac{1-3^7q+5^7q^2-7^7q^3+\&c.}{1-3q+5q^2-7q^3+\&c.}$$
$$=1-504\Sigma\sigma_5(n)\,q^n+5040\Sigma n\sigma_3(n)\,q^n-6720\Sigma n^2\sigma(n)\,q^n.$$

IV.

$$\frac{1-3^9q+5^9q^2-7^9q^3+\&c.}{1-3q+5q^2-7q^3+\&c.}$$
$$=1+480\Sigma\sigma_7(n)\,q^n-12096\Sigma n\sigma_5(n)\,q^n+72576\Sigma n^2\sigma_3(n)\,q^n$$
$$-80640\Sigma n^3\sigma(n)\,q^n.$$

V.

$$\frac{1-3^{11}q+5^{11}q^2-7^{11}q^3+\&c.}{1-3q+5q^2-7q^3+\&c.}$$
$$=1-264\Sigma\sigma_9(n)\,q^n+13200\Sigma n\sigma_7(n)\,q^n-190080\Sigma n^2\sigma_5(n)\,q^n$$
$$+887040\Sigma n^3\sigma_3(n)\,q^n-887040\Sigma n^4\sigma(n)\,q^n.$$

In all these formulæ the sign of summation refers to the letter n which is to have all values from 1 to ∞.

§ 24. The preceding results were obtained by the following process.

The values of the expressions

$$\frac{q^{\frac{1}{4}}-3^3q^{\frac{9}{4}}+5^3q^{\frac{25}{4}}-7^3q^{\frac{49}{4}}+\&c.}{q^{\frac{1}{4}}-3q^{\frac{9}{4}}+5q^{\frac{25}{4}}-7q^{\frac{49}{4}}+\&c.},$$

$$\frac{q^{\frac{1}{4}}-3^5q^{\frac{9}{4}}+5^5q^{\frac{25}{4}}-7^5q^{\frac{49}{4}}+\&c.}{q^{\frac{1}{4}}-3q^{\frac{9}{4}}+5q^{\frac{25}{4}}-7q^{\frac{49}{4}}+\&c.},$$

&c., &c.,

were expressed as symmetrical functions of the quantities I, G, E; or, more accurately, of the quantities i, g, e, where

$$i=\frac{4KI}{\pi^2},\ g=\frac{4KG}{\pi^2},\ e=\frac{4KE}{\pi^2}.^*$$

* This investigation, which is too lengthy for reproduction here, occurs in a work on the Zeta Functions which is still in the press.

I had previously investigated for another purpose the q-series which represent the homogeneous symmetrical functions of i, g, e, of the different forms, up to the fifth order,* and by replacing the above-mentioned symmetrical functions by their q-values the formulæ given in § 23 were obtained. These results seem to be of interest for their own sake, apart from the recurring formulæ to which they give rise, as they serve to express the quotients of

$$q - 3^3q^9 + 5^3q^{25} - 7^3q^{49} + \&c,$$
$$q - 3^5q^9 + 5^5q^{25} - 7^5q^{49} + \&c,$$
$$\&c., \qquad \&c.,$$

when divided by

$$q - 3q^9 + 5q^{25} - 7q^{49} + \&c.$$

The function E, § 25.

§ 25. The method just described also gives recurring formulæ, of a similar kind to those of § 2, in which the arguments are the same as in Euler's recurring formula

$$\sigma(n) - \sigma(n-1) - \sigma(n-2) + \sigma(n-5) + \sigma(n-7) - \&c. = 0,$$

where the numbers 1, 2, 5, 7, ... are the pentagonal numbers, and $\sigma(0)$ when it occurs is to have the value n.

If we use $E\phi(n)$ to denote the series

$$\phi(n) - \phi(n-1) - \phi(n-2) + \phi(n-5) + \phi(n-7)$$
$$- \phi(n-12) - \phi(n-15) + \&c.,$$

we may write Euler's result in the form

$$E\sigma(n) = 0.$$

Recurring formulæ expressed by means of E, §§ 26–29.

§ 26. Euler's formula is deducible from the analytical equation

$$\frac{q + 2q^2 - 5q^5 - 7q^7 + \&c.}{1 - q - q^2 + q^5 + q^7 - \&c.} = \Sigma\sigma(n)q^n,$$

and, by applying the method described in § 24, we find

$$\frac{q + 2^3q^2 - 5^3q^5 - 7^3q^7 + \&c.}{1 - q - q^2 + q^5 + q^7 - \&c.}$$
$$= -\tfrac{5}{12}\Sigma\sigma_3(n)q^n + \tfrac{3}{2}\Sigma n\sigma(n)q^n - \tfrac{1}{12}\Sigma\sigma(n)q^n.$$

* These values are given in the next paper (p. 65).

§ 27. This equation gives the formula

$$5E\sigma_3(n) - 18E\{n\sigma(n)\} = 0,$$

where $\sigma_3(0)$, when it occurs, is to have the value $-\frac{1}{5}(12n^2+n)$.
Writing the series at full length, this formula is

$$5\{\sigma_3(n) - \sigma_3(n-1) - \sigma_3(n-2) + \sigma_3(n-5) + \sigma_3(n-7) - \&c.\}$$

$$-18\{n\sigma(n) - (n-1)\sigma(n-1) - (n-2)\sigma(n-2) + (n-5)\sigma(n-5) + \&c.\} = 0.$$

As examples, putting $n = 3$ and 5,

$$5\{\sigma_3(3) - \sigma_3(2) - \sigma_3(1)\} - 18\{3\sigma(3) - 2\sigma(2) - \sigma(1)\} = 0,$$

and $5\{\sigma_3(5) - \sigma_3(4) - \sigma_3(3) + \sigma_3(0)\}$

$$- 18\{5\sigma(5) - 4\sigma(4) - 3\sigma(3)\} = 0,$$

where $\sigma_3(0) = -\frac{1}{5} \times 305 = -61.$

These equations give

$$5\{28 - 9 - 1\} - 18\{12 - 6 - 1) = 0,$$

and $5\{126 - 73 - 28 - 61\} - 18\{30 - 28 - 12\} = 0,$

which are evidently true.

§ 28. By substituting for $E\sigma(n)$ its value from Euler's formula, we may deduce the relation

$$\sigma_3(n) - \sigma_3(n-1) - \sigma_3(n-2) + \sigma_3(n-5) + \sigma_3(n-7) + \&c.$$

$$- \tfrac{18}{5}\{\sigma(n-1) + 2\sigma(n-2) - 5\sigma(n-5) - 7\sigma(n-7) + \&c.\} = 0.$$

where $\sigma(0)$ and $\sigma_3(0)$, when they occur, are to have respectively the values $\frac{1}{3}n$ and $-\frac{1}{5}n$.

§ 29. In the same manner, we may obtain formulæ connecting $E\sigma_5(n)$, $E\{n\sigma_3(n)\}$, $E\{n^2\sigma(n)\}$, &c. corresponding to the J-formulæ of § 2.