Let $\alpha_1, \beta_1, \gamma_1, \alpha_2, \beta_2, \gamma_2, \ldots, \alpha_6, \beta, \gamma_6$
be the coordinates of the lines, forming the hexagon of reference; then the invariant, called B in Salmon's *Higher Plane Curves* is the square of the determinant

$$\begin{vmatrix} \alpha_1^2, & \alpha_2^2, & \alpha_3^2, & \alpha_4^2, & \alpha_5^2, & \alpha_6^2 \\ \alpha_1\beta_1, & \alpha_2\beta_2, & \ldots & \ldots & \ldots & \ldots \\ \alpha_1\gamma_1, & \alpha_2\gamma_2, & \ldots & \ldots & \ldots & \ldots \\ \beta_1^2, & \beta_2^2, & \ldots & \ldots & \ldots & \ldots \\ \beta_1\gamma_1, & \beta_2\gamma_2, & \ldots & \ldots & \ldots & \ldots \\ \gamma_1^2, & \gamma_2^2, & \ldots & \ldots & \ldots & \ldots \end{vmatrix}.$$

This invariant (the catalecticant of the quartic) vanishes then, not only if the quartic can be expressed as the sum of five fourth powers but, also if the sides of the hexagon of reference touch a conic, and this includes the first theorem.

The corresponding theorem for a quartic surface is that its catalecticant vanishes, if the faces of the dekahedron of reference touch a quadric.

The curve of the sixth degree can be expressed as the sum of ten sixth powers; and if the sides of this decagon touch a curve of the third class, the catalecticant of the sextic vanishes.

ON THE APPLICATION OF ABEL'S THEOREM TO ELLIPTIC INTEGRALS OF THE FIRST KIND.

By *W. Burnside.*

Take as the fixed curve for an application of Abel's theorem the quartic

$$y^2 = \frac{ax^2+2bx+c}{Ax^2+2Bx+C} = \frac{N_x}{D_x} \text{ say } \ldots\ldots\ldots\ldots\text{(i)},$$

and for the variable curve the hyperbola

$$y = \frac{mx+n}{m'x+n'}, \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(ii)}.$$

Then if $N_xD_x = X$, a general quartic function,

$$\sum_1^4 \frac{dx_r}{\sqrt{X_r}} = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(iii)},$$

where x_1, x_2, x_3, x_4 are the abscissæ of the points of intersection of (i) and (ii), when the constants m, n, m', n' vary in any way.

If y_1, y_2, y_3, y_4 are the corresponding ordinates, the elimination of m, n, m', n' between the four equations

$$y_r = \frac{mx_r + n}{m'x_r + n'} \qquad r = 1, 2, 3, 4,$$

shews that the cross-ratio of the x's is equal to the cross-ratio of the corresponding y's, so that

$$\frac{(y_1 - y_3)(y_2 - y_4)}{(y_1 - y_2)(y_3 - y_4)} = \frac{(x_1 - x_3)(x_2 - x_4)}{(x_1 - x_2)(x_3 - x_4)} \text{........(iv)},$$

is the form in which the integral relation corresponding to the differential relation (iii) immediately offers itself.

It is my object to exhibit some of the equations into which (iv) may be algebraically transformed by using (i).

The two values of x which correspond to the same value of y are connected by a linear substitution: for if

$$y^2 = \frac{ax^2 + 2bx + c}{Ax^2 + 2Bx + C} = \frac{ax'^2 + 2Bx' + c}{Ax'^2 + 2Bx' + C},$$

then $2(aB - bA)xx' + (aC - cA)(x + x') + 2(bC - cB) = 0$,

or
$$x' = \frac{x - p}{qx - 1},$$

where
$$p = 2\frac{bC - cB}{cA - aC}, \quad q = 2\frac{aB - bA}{cA - aC}.$$

Hence,
$$y^2 - y_0^2 = \frac{a - Ay_0^2}{D_x}(x - x_0)(x - x'_0)$$

$$= (cA - aC)\frac{(x - x_0)(qxx_0 - x - x_0 + p)}{D_x D_0} \text{...(v)},$$

and
$$\frac{y - y_0}{x - x_0} = \frac{a - Ay_0^2}{D_x}\,\frac{x - x_0'}{y + y_0} \text{..................(v)}'.$$

If now equation (iv) be written in the form

$$\frac{y_1 - y_2}{x_1 - x_2} \cdot \frac{x_1 - x_3}{y_1 - y_3} = \frac{y_4 - y_2}{x_4 - x_2} \cdot \frac{x_4 - x_3}{y_4 - y_3},$$

and the transformation (v)′ be applied to each side, there results at once

$$\frac{x_1 - x_2'}{y_1 + y_2} \cdot \frac{y_1 + y_3}{x_1 - x_3'} = \frac{x_4 - x_2'}{y_4 + y_2} \cdot \frac{y_4 + y_3}{x_4 - x_3'},$$

or
$$\frac{(y_1 + y_3)(y_2 + y_4)}{(y_1 + y_2)(y_3 + y_4)} = \frac{(x_1 - x_3')(x_2' - x_4)}{(x_1 - x_2')(x_3' - x_4)} \text{.........(vi)}.$$

Subtracting unity from each side of this equation, it becomes

$$\frac{(y_1-y_4)(y_2-y_3)}{(y_1+y_2)(y_3+y_4)}=\frac{(x_1-x_4)(x_2'-x_3')}{(x_1-x_2')(x_3'-x_4)},$$

while (iv) similarly gives

$$\frac{(y_1-y_4)(y_2-y_3)}{(y_1-y_2)(y_3-y_4)}=\frac{(x_1-x_4)(x_2-x_3)}{(x_1-x_2)(x_3-x_4)}.$$

Dividing the last two equations one by the other

$$\frac{(y_1-y_2)(y_3-y_4)}{(y_1+y_2)(y_3+y_4)}=\frac{x_2'-x_3'}{x_2-x_3}\cdot\frac{(x_1-x_2)(x_3-x_4)}{(x_1-x_2')(x_3'-x_1)}$$

$$=(1-pq)\frac{(x_1-x_2)(x_3-x_4)}{(qx_1x_2-x_1-x_2+p)(qx_3x_4-x_3-x_4+p)}\;\text{....(vii)},$$

and now the terms involving the coordinates of points 1 and 2 are separated from these involving 3 and 4.

Since, from (v),

$$y_1^2-y_2^2=\frac{cA-aC}{D_1D_2}(x_1-x_2)(qx_1x_2-x_1-x_2+p),$$

equation (vii) may be written in the simple form

$$\left[\frac{(y_1-y_2)(y_3-y_4)}{(x_1-x_2)(x_3-x_4)}\right]^2 D_1D_2D_3D_4=(cA-aC)^2(1-pq)$$

$$=(cA-aC)^2-4(aB-bA)(bC-cB)\;\text{...(viii)},$$

or

$$\left[\frac{(\sqrt{N_1}\sqrt{D_2}-\sqrt{N_2}\sqrt{D_1})(\sqrt{N_3}\sqrt{D_4}-\sqrt{N_4}\sqrt{D_3})}{(x_1-x_2)(x_3-x_4)}\right]^2$$

$$=(cA-aC)^2-4(aB-bA)(bC-cB).$$

The transformations thus obtained of the original integral relation (iv) by direct algebraical process are more easily derived by considering those substitutions which transform both the equation (i) and the differential

$$\frac{dx}{\sqrt{X}}$$

respectively into themselves.

To illustrate this I take equation (i) in the standard form

$$y^2=\frac{1-x^2}{1-k^2x^2}\;\text{.........................(i)}',$$

to which it can always be reduced by a linear substitution performed on x.

The three sets of substitutions

$$x=-x', \quad y=-y' \ldots\ldots\ldots\ldots\ldots\ldots (A);$$

$$x=\frac{1}{kx'}, \quad y=\frac{1}{ky'} \ldots\ldots\ldots\ldots\ldots\ldots (B);$$

$$x=-\frac{1}{kx'}, \quad y=-\frac{1}{ky'}, \ldots\ldots\ldots\ldots\ldots\ldots (C);$$

transform both the equation (i)′ and the differential

$$\frac{dx}{(1-k^2x^2)\,y}\left(=\frac{dx}{\sqrt{(1-x^2.1-k^2x^2)}}\right)$$

into themselves.

Hence, if $f(x_1, y_1, x_2, y_2, x_3, y_3, x_4, y_4)=0$ is an integral relation corresponding to the differential relation

$$\sum_1^4 \frac{dx_r}{(1-k^2x_r^2)\,y_r}=0,$$

a new integral relation may be obtained by transforming one or more of the pairs x_r, y_r in $f=0$ by any one of the above substitutions. This new integral relation will not of course necessarily correspond to the same value of the constant of integration. Starting again from the original integral form

$$\frac{y_1-y_2.y_3-y_4}{x_1-x_2.x_3-x_4}=\frac{y_1-y_3.y_2-y_4}{x_1-x_3.x_2-x_4},$$

the substitution (A) applied to the terms with suffix 3 and 4 gives

$$\frac{y_1-y_2.y_3-y_4}{x_1-x_2.x_3-x_4}=\frac{y_1+y_3.y_2+y_4}{x_1+x_3.x_2+x_4},$$

which is equivalent to the preceding equation, since each are satisfied by the set of values

$$x_1=x_4=0,\ y_1=y_4=1,\ x_2=x_3=1,\ y_2=y_3=0.$$

Adding and subtracting the numerators and denominators of the right hand sides, there results

$$\frac{y_1-y_2.y_3-y_4}{x_1-x_2.x_3-x_4}=\frac{y_1y_3+y_2y_4}{x_1x_3+x_2x_4}=\frac{y_1y_4+y_2y_3}{x_1x_4+x_2x_3},$$

and from their mode of formation all three results remain true when the suffixes are permuted in any order.

Using the substitution (B) for the terms with suffix 3 and 4 in the equations

$$\frac{y_1 - y_2 \cdot y_3 - y_4}{x_1 - x_2 \cdot x_3 - x_4} = \frac{y_1 \pm y_3 \cdot y_2 \pm y_4}{x_1 \pm x_3 \cdot x_2 \pm x_4} = \frac{y_1 y_2 + y_3 y_4}{x_1 x_2 + x_3 x_4},$$

it follows on casting out the common facter $x_3 x_4 / y_3 y_4$, that

$$\frac{y_1 - y_2 \cdot y_3 - y_4}{x_1 - x_2 \cdot x_3 - x_4} = \frac{1 \pm k y_1 y_3 \cdot 1 \pm k y_2 y_4}{1 \pm k x_1 x_3 \cdot 1 \pm k x_2 x_4} = \frac{1 + k^2 y_1 y_2 y_3 y_4}{1 + k^2 x_1 x_2 x_3 x_4},$$

these equations being still equivalent to the preceding; and again the suffixes may be permuted in any way. In particular, the equation

$$\frac{\Sigma y_r y_s}{\Sigma x_r x_s} = \frac{1 + k^2 y_1 y_2 y_3 y_4}{1 + k^2 x_1 x_2 x_3 x_4}$$

involving only symmetric functions is a form of the integral equation.

The right-hand side of the general form (viii) becomes in this case $(1 - k^2)^2$ or k'^4; and hence the square of any one of the quantities of which

$$\frac{y_1 - y_2 \cdot y_3 - y_4}{x_1 - x_2 \cdot x_3 - x_4}, \quad \frac{y_1 + y_2 \cdot y_3 + y_4}{x_1 + x_2 \cdot x_3 + x_4},$$

$$\frac{1 - k y_1 y_2 \cdot 1 - k y_3 y_4}{1 - k x_1 x_2 \cdot 1 - k x_3 x_4}, \quad \frac{1 + k y_1 y_2 \cdot 1 + k y_3 y_4}{1 + k x_1 x_2 \cdot 1 + k x_3 x_4},$$

$$\frac{y_1 y_2 + y_3 y_4}{x_1 x_2 + x_3 x_4}, \quad \frac{1 + k^2 y_1 y_2 y_3 y_4}{1 + k^2 x_1 x_2 x_3 x_4},$$

are types, is equal to

$$\frac{k'^4}{1 - k^2 x_1^2 \cdot 1 - k^2 x_2^2 \cdot 1 - k^2 x_3^2 \cdot 1 - k_2 x_4^2},$$

or

$$\frac{1 - k^2 y_1^2 \cdot 1 - k^2 y_2^2 \cdot 1 - k^2 y_3^2 \cdot 1 - k^2 y_4^2}{k'^4}.$$

If

$$\frac{dx_r}{(1 - k^2 x_r) y_r} = du_r,$$

so that (i)′ becomes

$$u_1 + u_2 + u_3 + u_4 \equiv \text{constant},$$

the value of the constant for the set of equivalent integrals just given is $2K$; for, taking for example the integral

$$\left(\frac{1 + k^2 y_1 y_2 y_3 y_4}{1 + k^2 x_1 x_2 x_3 x_4}\right)^2 = \frac{k'^4}{D_1 D_2 D_3 D_4},$$

it is satisfied by the values

$$x_1 = 1,\quad x_2 = 1,\quad x_3 = 0,\quad x_4 = 0;$$
$$y_1 = 0,\quad y_2 = 0,\quad y_3 = 1,\quad y_4 = 1,$$

and the differential relation is therefore

$$\int_1^{x_1} du_1 + \int_1^{x_2} du_2 + \int_0^{x_3} du_3 + \int_0^{x_4} du_4 \equiv 0,$$

or $$\int_0^{x_1} du_1 + \int_0^{x_2} du_2 + \int_0^{x_3} du_3 + \int_0^{x_4} du_4 \equiv 2\int_0^1 du \equiv 2K.$$

The equations in question then give in different forms some of the relations between Jacobian elliptic functions of four arguments whose sum is $2K$.

In particular the relation which has just been used to determine the constant becomes

$$\operatorname{dn} u_1 \operatorname{dn} u_2 \operatorname{dn} u_3 \operatorname{dn} u_4 + k^2 \operatorname{cn} u_1 \operatorname{cn} u_2 \operatorname{cn} u_3 \operatorname{cn} u_4$$
$$= k'^2 \left[1 + k^2 \operatorname{sn} u_1 \operatorname{sn} u_2 \operatorname{sn} u_3 \operatorname{sn} u_4\right],$$

the sign of the square root that has been taken being determined from any set of special values. Writing $u_4 + 2K$ for u_4, this is the relation due to Professor Cayley.

The other symmetrical equation in this case is

$$\Sigma \operatorname{cn} u_p \operatorname{cn} u_q \operatorname{dn} u_r \operatorname{dn} u_s = k'^2 \Sigma \operatorname{sn} u_p \operatorname{sn} u_q.$$

The number of forms into which the integral equation may be thrown, even for the same value of the constant, is practically endless.

Thus, from

$$\frac{y_1 y_2 + y_3 y_4}{x_1 x_2 + x_3 x_4} = \frac{y_1 y_3 + y_2 y_4}{x_1 x_3 + x_2 x_4},$$

on multiplying up and re-arranging the terms, there results

$$\frac{x_1 y_1 - x_4 y_4}{x_1 y_4 - x_4 y_1} + \frac{x_2 y_2 - x_3 y_3}{x_2 y_3 - x_3 y_2} = 0;$$

while from

$$\frac{y_1 y_4 + y_2 y_3}{x_1 x_4 + x_2 x_3} = \frac{1 + k^2 y_1 y_2 y_3 y_4}{1 + k^4 x_1 x_2 x_3 x_4},$$

it follows in the same way that

$$\frac{1 - k^2 x_1 x_4 y_1 y_4}{y_1 y_4 - x_1 x_4} + \frac{1 - k^2 x_2 x_3 y_2 y_3}{y_2 y_3 - x_2 x_3} = 0,$$

and in each of these new equations the suffixes are separated into two sets.

If one of the substitutions be applied to a single set of terms in any one of the equivalent equations already obtained, say to x_4, y_4, a new integral relation corresponding to a different value of the constant of integration will result.

Thus applying the substitution (B) to the terms x_4, y_4 in values of the equations first used, it follows that

$$\frac{y_1 + ky_2y_3y_4}{x_1 + kx_2x_3x_4} = \frac{y_4 + ky_1y_2y_3}{x_4 + kx_1x_2x_3}.$$

This equation may also be written

$$\frac{x_1y_4 - x_4y_1}{x_1y_1 - x_4y_4} + \frac{k(y_2y_3 - x_2x_3)}{1 - k^2x_2x_3y_2y_3} = 0.$$

Since $x_1 = x_2 = x_3 = 0$, $y_1 = y_2 = y_3 = 1$, $x_4 = \infty$, $y_4 = \frac{-1}{\sqrt{k}}$ are a special set of values satisfying this equation, the sum of four corresponding arguments $u_1 + u_2 + u_3 + u_4$ is $2K + iK'$.

ON THE LINEAR TRANSFORMATION OF THE ELLIPTIC DIFFERENTIAL.

By *W. Burnside.*

THE purpose of this paper is to exhibit in as simple and systematic form as possible the more important of the linear transformations of an elliptic integral of the first kind. I shall consider in order the transformation of the integral (i) into itself, (ii) into Weierstrass's normal form, (iii) into Legendre's, and (iv) into Riemann's normal form.

A single linear substitution can always be found which will transform three arbitrarily given values of a variable quantity into three other arbitrarily given values, each into