Again it is shewn in the same Appendix that

$$\frac{d^n z}{dy^n} = (\theta^{-n}) \frac{d^n z}{dx^n} + \frac{n-1}{1} \frac{d\theta^{-n}}{dh} \frac{d^{n-1}z}{dx^{n-1}} + \frac{n-1.n-2}{1.2} \frac{d^2\theta^{-n}}{dh^2} \frac{d^{n-2}z}{dx^{n-2}} + \dots$$

$$\dots + \frac{d^{n-1}(\theta^{-n})}{dh^{n-1}} \frac{dz}{dx},$$

and hence we get

$$\frac{d^n z}{dy^n} = \frac{(-)^{n-1}}{\phi'^{\,2n-1}}$$

$$\times \begin{vmatrix} \frac{n}{2!}\phi'' , & \frac{1}{1!}\phi' & & \\ \frac{2n}{3!}\phi''' , & \frac{n+1}{2!}\phi'' , & \frac{2}{1!}\phi' & \\ \frac{3n}{4}\phi^{\text{iv}} , & \frac{2n+1}{3!}\phi''' , & \frac{n+2}{2!}\phi'' & , \dots \\ \vdots & \vdots & \vdots & \\ \frac{n-1\,n}{n!}\phi^{(n)}, & \frac{(n-2)\,n+1}{n-1!}\phi^{(n-1)}, & \frac{(n-3)\,n+2}{n-2!}\phi^{(n-2)}, & \dots \\ \frac{1}{n-1!}z^{(n)} , & \frac{1}{n-2!}z^{(n-1)} , & \frac{1}{n-3!}z^{(n-2)} & , \dots \end{vmatrix}.$$

ON THE CIRCLE PERPEND-FEET PENCIL AND ORTHOCENTRAL AXIS OF A COMPLETE QUADRILATERAL.

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[*References to Casey's* "*Sequel to Euclid.*"]

I. THE nine-points circle of a triangle ABC is the locus of the middle point I of a vector from the orthocentre X to the circum-circle.

Dem. The nine-points circle is known to bisect the vectors XA, XB, XC. Hence X is the centre of similitude of the circum-circle and nine-points circle. Hence if XO be a vector to any point O on the circum-circle, its middle point I traces the nine-points circle.

II. The nine-points circle of a triangle ABC is the locus of the intersection I of a vector XO from the orthocentre X to any point O on the circum-circle with the line EFG of collinearity of the feet E, F, G of the three perpends OE, OF, OG from the point O upon the sides a, b, c.

Dem. This follows from Casey's *Sequel*, III. 14, wherein the intersection I is shown to be the middle point of XO.

III. Produce the three perpends AX, BX, CX of the triangle ABC to meet the circum-circle again in points α, β, γ. Let the vectors $O\alpha$, $O\beta$, $O\gamma$ from any point O on the circum-circle cut the sides a, b, c respectively in e, f, g. Then e, f, g, lie in a straight line passing through X, parallel to EFG and at twice the distance from O.

Dem. This follows from Casey's *Sequel*, III. 14, wherein it is shown (as regards the side c) that Xg is parallel to EFG, and at twice the distance from O. Similarly, Xe, Xf are parallel to EFG, so that e, f, g, X are in one straight line parallel to EFG, at twice the distance from O.

The three points α, β, γ will be called the *circle perpend-feet.*

The above properties may now be extended to quadrilaterals.

Preliminary. a, b, c, d are four straight lines, forming a complete quadrilateral, and forming also the four triangles bcd, cda, dab, abc. The four circum-circles of these triangles are known to meet in a point, say O, which may be called the *common circum-circle point* (Casey's *Sequel*, III. 12, Cor. 3).

Let E, F, G, H be the feet of perpends from O upon a, b, c, d; and let X, Y, Z, W be the orthocentres of the four triangles. Then $EFGH$, $XYZW$ are known to be two parallel straight lines, and $EFGH$ is at half the distance of $XYZW$ from O (Casey's *Sequel*, III. 12, Cor. 2, and III. 14, Cor.). The line $EFGH$ may be called the *axis of perpend-feet*, and the line $XYZW$ may be called the *orthocentral axis.*

IV. The axis of perpend-feet $EFGH$ of a complete quadrilateral $abcd$ passes through the middle points x, y, z, w of the vectors OX, OY, OZ, OW from O to the orthocentres X, Y, Z, W of the four triangles bcd, cda, dab, abc, and these points x, y, z, w lie on the nine-points circles of the four triangles.

Dem. This is evident from Theorem II.

V. Produce the twelve perpends of the above four triangles *bcd*, *cda*, *dab*, *abc*, each to cut the circum-circle of its own triangle again in the twelve points styled "circle perpend-feet," which may be denoted as below.

Triangle.	CIRCLE PERPEND-FEET upon perpends to side			
	a	*b*	*c*	*d*
bcd	.	β_1	γ_1	δ_1
acd	α_2	.	γ_2	δ_2
abd	α_3	β_3	.	δ_3
abc	α_4	β_4	γ_4	.

The notation shows that the three points α_2, α_3, α_4 are the intersections of the three perpends upon the side *a*, of the three triangles *acd*, *abd*, *abc* with their circum-circles. Similarly of the rest.

Then α_2, α_3, α_4 lie in one straight line through *O*.
" β_1, β_3, β_4 " " "
" γ_1, γ_2, γ_4 " " "
" δ_1, δ_2, δ_3 " " "

Also these four lines cut the lines *a*, *b*, *c*, *d* respectively in four points *e*, *f*, *g*, *h*, lying on the axis *XYZW*.

Dem. By Theorem III., the three vectors $O\alpha_2$, $O\gamma_2$, $O\delta_2$ from *O* to the "circle perpend-feet" α_2, γ_2, δ_2 of the triangle *acd* cut the sides *a*, *c*, *d* respectively of that triangle in three points, say e'', g'', h'', which lie on a straight line through *Y* parallel to *EGH*, *i.e.* on the line *XZW*.

Similarly the three vectors $O\alpha_3$, $O\beta_3$, $O\delta_3$ cut the sides *a*, *b*, *d* of the triangle *abd* in three points, say e''', f''', h''', which lie on a straight line, through *Z* parallel to *EFH*, *i.e.* on the line *XYW*. Likewise the three vectors $O\alpha_4$, $O\beta_4$, $O\gamma_4$ cut the sides *a*, *b*, *c* of the triangle *abc* in three points, say e^{iv}, f^{iv}, g^{iv}, which lie on a line through *W* parallel to *EFG*, *i.e.* on the line *XYZ*.

Thus the three lines $O\alpha_2$, $O\alpha_3$, $O\alpha_4$ cut the line *a* at its intersections e'', e''', e^{iv} with the three lines *XZW*, *XYW*, *XYZ*; but these last form one line *XYZW*. Thus the three points e'', e''', e^{iv} coincide in one point, say *e*, the intersection of *a* with *XYZW*; and the lines $O\alpha_2$, $O\alpha_3$, $O\alpha_4$ form one straight line.

Similarly *O*, β_2, β_3, β_4 are in one straight line cutting *b* in a point *f* which lies on *XYZW*; and so on.

These properties may be expressed in the following theorem:—

The twelve circle perpend-feet $(\beta_1, \gamma_1, \delta_1)$, $(\alpha_2, \gamma_2, \delta_2)$, $(\alpha_3, \beta_3, \delta_3)$, $(\alpha_4, \beta_4, \gamma_4)$ of the four triangles *bcd*, *acd*, *abd*, *abc* formed by the sides of a complete quadrilateral lie by threes upon four lines $O\alpha_2\alpha_3\alpha_4$, $O\beta_1\beta_3\beta_4$, $O\gamma_1\gamma_2\gamma_4$, $O\delta_1\delta_2\delta_3$ forming a pencil whose vertex is the common circum-circle point O; and these four lines cut the sides a, b, c, d respectively in the points e, f, g, h all lying upon the orthocentral axis $XYZW$.

NOTE ON A REMARKABLE SERIES OF NUMBERS.

By *H. W. Segar.*

The differences of the numbers

$$1, 2, 5, 13, 34, 89, 233 \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

are

$$1, 3, 8, 21, 55, 144, 377 \ldots\ldots\ldots\ldots\ldots\ldots (2).$$

Let u_1, u_2, u_3, ... denote the terms of the first, and v_1, v_2, v_3, ... those of the second series.

The series (1) is the series of least numbers satisfying

$$\frac{u_1}{u_2} > \frac{u_2}{u_3} > \frac{u_3}{u_4} > \ldots,$$

and (2) is the series of least numbers satisfying

$$\frac{v_1}{v_2} < \frac{v_2}{v_3} < \frac{v_3}{v_4} < \ldots .$$

Between the terms of the two series there exist the following relations

$$\begin{aligned} u_{2n} &= u_n^2 + v_n^2, \\ u_{2n+1} &= u_{n+1}^2 + v_n^2, \\ v_{2n} &= u_{n+1}^2 - u_n^2, \\ v_{2n+1} &= v_{n+1}^2 - v_n^2; \end{aligned}$$

these can be verified by substituting the values

$$u_n = \frac{\sqrt{5}+1}{2\sqrt{5}}\left(\frac{3+\sqrt{5}}{2}\right)^{n-1} + \frac{\sqrt{5}-1}{2\sqrt{5}}\left(\frac{3-\sqrt{5}}{2}\right)^{n-1},$$

and $$v_n = \frac{1}{\sqrt{5}}\left\{\left(\frac{3+\sqrt{5}}{2}\right)^n - \left(\frac{3-\sqrt{5}}{2}\right)^n\right\},$$
