MESSENGER OF MATHEMATICS.

ON A CLASS OF FUNCTIONS DERIVABLE FROM THE COMPLETE ELLIPTIC INTEGRALS, AND CONNECTED WITH LEGENDRE'S FUNCTIONS.

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I.

Theory of associated functions.

Transformation of the linear differential equation of the second order, § 1.

§ 1. Consider the equation

$$N\frac{d^2y}{dx^2}+P\frac{dy}{dx}+Qy=0 \quad \text{..............(1).}$$

If in this equation $N=0$, it reduces to an equation of the first order.

If $Q=0$, the substitution $\dfrac{dy}{dx}=z$ reduces this equation to one of the first order in z.

We shall suppose in the future that neither N nor Q are equal to 0.

This differential equation may be transformed to the following form:

$$\frac{d}{dx}\left\{\frac{1}{\Lambda}\frac{dy}{dx}\right\}=Ly \quad \text{..................(2),}$$

where Λ and L are some functions of x, which may be found, when N, P, and Q are given, by means of algebraical operations and one integration.

For, developing the expression in brackets in (2), we have

$$\frac{1}{\Lambda}\frac{d^2y}{dx^2}-\frac{1}{\Lambda^2}\frac{d\Lambda}{dx}\frac{dy}{dx}-Ly=0;$$

whence, by comparison with the given equation (1), we find

$$\Lambda = Ce^{-\int \frac{P}{N} dx} \quad \text{..........................(3)},$$

$$L = -\frac{Q}{N\Lambda} = -\frac{1}{C}\frac{Q}{N}e^{\int \frac{P}{N} dx} \quad \text{............(4)},$$

C being an arbitrary constant. Again, if Λ and L be given, we have

$$N : P : Q = 1 : -\frac{d \log \Lambda}{dx} : -\Lambda L \text{............(5)}.$$

The associated function η, § 2.

§ 2. We introduce a new function η, connected with y by the differential relation

$$\frac{1}{\Lambda}\frac{dy}{dx} = \eta.$$

The equation (2) gives then

$$\frac{d\eta}{dx} = Ly.$$

Thus y and η are mutually connected by the symmetric equations

$$\frac{dy}{dx} = \Lambda\eta \text{ ....(6)},$$

$$\frac{d\eta}{dx} = Ly \text{ ....(7)}.$$

Differential equation satisfied by the associated function, § 3.

§ 3. Differentiating (7) and substituting in the right-hand side of the result of differentiation for y and $\frac{dy}{dx}$ their values expressed in $\frac{dy}{dx}$ and η from (6) and (7), we find the following differential equation of the second order satisfied by the function η,

$$\frac{1}{L}\frac{d^2\eta}{dx^2} - \frac{1}{L^2}\frac{d\eta}{dx}\frac{dL}{dx} - \Lambda\eta = 0,$$

or

$$N_1\frac{d^2\eta}{dx^2} + P_1\frac{d\eta}{dx} + Q_1\eta = 0 \text{ ....(8)},$$

where the new functions N_1, P_1, Q_1 are

$$N_1 : P_1 : Q_1 = 1 : -\frac{d \log L}{dx} : -L\Lambda,$$

or else

$$L = C_1 e^{-\int \frac{P_1}{N_1} dx},$$

$$\Lambda = -\frac{Q_1}{N_1 L} = -\frac{1}{C_1}\frac{Q_1}{N_1} e^{\int \frac{P_1}{N_1} dx},$$

C_1 being a new arbitrary constant. It will be noticed that the equations of this section are in strict analogy with those of § 1.

The relation of association, § 4.

§ 4. The differential equation (8) may be presented in the form

$$\frac{d}{dx}\left\{\frac{1}{L}\frac{d\eta}{dx}\right\} = \Lambda\eta \text{ ....(9)}$$

in which it appears to be analogous to (2). We shall call the function η *associated to* y. It is obvious that *to every function satisfying a linear differential equation of the form* (2) *corresponds an associated function, satisfying a differential equation of the same form;* also that *there is only one such function;* and that the relation of association is mutual, *i.e. if* η *is associated to* y, y *is associated to* η. We shall therefore say in future that η is associated *with* y.

Examples of associated functions, § 5.

§ 5. The simplest case of associated functions is presented by the circular functions sin and cos. For we have

$$\frac{d}{d\theta}\sin\theta = +\cos\theta,$$

$$\frac{d}{d\theta}\cos\theta = -\sin\theta.$$

Thus, if we put $\quad y = a \sin c\theta$ ....(10),

$\qquad\qquad\qquad\quad \eta = a \cos c\theta$ ....(11),

we shall have
$$\frac{dy}{d\theta} = c\eta,$$
$$\frac{d\eta}{d\theta} = -cy,$$
i.e.
$$\Lambda = c, \quad L = -c.$$

Another simple case of associated functions is presented by the hyperbolic sine and cosine. Put generally

$$y = \tfrac{1}{2}a\,(e^{c\theta} + e^{-c\theta}) \quad \text{..................(12)},$$
$$\eta = \tfrac{1}{2}a\,(e^{c\theta} - e^{-c\theta}) \quad \text{..................(13)}.$$

Then we have
$$\frac{dy}{d\theta} = c\eta,$$
$$\frac{d\eta}{d\theta} = cy.$$

Thus, here
$$\Lambda = c, \quad L = c.$$

As a third example of associated functions, we take the complete elliptic integrals K and E, where

$$K = \int_0^{\frac{1}{2}\pi} \frac{d\theta}{\sqrt{(1 - k^2 \sin^2\theta)}} \quad \text{...............(14)},$$
$$E = \int_0^{\frac{1}{2}\pi} \sqrt{(1 - k^2 \sin^2\theta)}\, d\theta \quad \text{...........(15)}.$$

Considering these as functions of k, we have

$$\frac{dK}{dk} = \frac{G}{kk'^2} \quad \text{......................(16)},$$
$$\frac{dG}{dk} = kK \quad \text{......................(17)},$$

where we have put, with Dr. Glaisher,

$$G = E - k'^2 k \quad \text{.....................(18)}.$$

Thus K and G are associated functions, and we have in this case

$$\Lambda = \frac{1}{ckk'^2}, \quad L = ck.$$

At the same time we have

$$\frac{dE}{dk} = \frac{I}{k} \quad \text{.........................(19)},$$
$$\frac{dI}{dk} = -\frac{k}{k'^2}E \quad \text{.....................(20)},$$

where
$$I = E - K \quad \text{........................(21)}.$$

Thus E and I are associated functions, and for them

$$\Lambda = \frac{1}{ck},\ L = -\frac{ck}{k'^2}.$$

Dr. Glaisher has introduced in some of his investigations* (besides G and I) another new combination of the elliptic integrals K and E, which he denotes by W, viz.

$$W = \tfrac{1}{2}(I + E) \text{ ....}(22),$$

Again, I have considered in a paper published in the *Messenger* in 1889 the quantity H defined by the equation

$$H = Ek^2 - Ik'^2 \text{ ....}(23).$$

By differentiation of W and H we find easily the following relations

$$\frac{dW}{dk} = -\frac{H}{2kk'^2} \text{ ....}(24),$$

$$\frac{dH}{dk} = \ \ 6kW \text{ ....}(25).$$

Thus W and N are also associated functions, where

$$\Lambda = -\frac{1}{2kk'^2},\ L = 6ck.$$

New form of the differential equation and the relations of association, § 6.

§ 6. It may easily be shown that, by the change of the independent variable only, the quantities Λ, L may be transformed so as to satisfy the relation

$$\Lambda . L = -c,$$

c being any constant. For, taking

$$z = \phi(x),$$

we have

$$\frac{d}{dx} = \phi'(x)\frac{d}{dz},$$

$$\frac{d^2}{dx^2} = [\phi'(x)]^2 \frac{d^2}{dz^2} + \phi''(x)\frac{d}{dz},$$

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* "On the quantities K, E, J, G, K', E', J', G' in Elliptic Functions," *Quarterly Journal of Mathematics*, Vol. xx. No. 80. The quotations below refer to this paper.

and thus equation (1) becomes

$$N[\phi'(x)]^2 \frac{d^2y}{dz^2} + [N\phi''(x) + P\phi'(x)] \frac{dy}{dz} + Qy = 0,$$

or
$$N_1 \frac{d^2y}{dz^2} + P_1 \frac{dy}{dz} + Q_1 y = 0\,,$$

N_1, P_1, Q_1, being the results of the substitution of z instead of x in the differential equation. Thus, from the relation

$$L\Lambda = -\frac{Q_1}{N_1}\,,$$

we have, if $L\Lambda$ must be equal to $-c$,

$$\frac{Q}{N[\phi'(x)]^2} = +c,$$

or
$$z = \phi(x) = \int \left(\frac{Q}{cN}\right)^{\frac{1}{2}} dx \quad \text{...............(26)},$$

Q. E. F.

In the example of the preceding section, we have for the elliptic integrals

$$z = \phi(k) = \int \frac{dk}{k'} = \int \frac{dk}{\sqrt{(1-k^2)}} = \text{arc} \sin k.$$

Thus, if we take the modular angle θ as the independent variable, we shall have $\Lambda L = c$, and in fact we find

$$\frac{dK}{d\theta} = \quad 2G \operatorname{cec} 2\theta \quad \text{..................(27)},$$

$$\frac{dG}{d\theta} = \quad \tfrac{1}{2} K \sin 2\theta \quad \text{..................(28)},$$

$$\frac{dE}{d\theta} = \quad I \cot \theta \quad \text{..................(29)},$$

$$\frac{dI}{d\theta} = - \quad E \tan \theta \quad \text{..................(30)},$$

$$\frac{dW}{d\theta} = - \quad H \operatorname{cec} 2\theta \quad \text{..................(31)},$$

$$\frac{dH}{d\theta} = \quad 3W \sin 2\theta \quad \text{..................(32)}.$$

Hypergeometric functions, § 7.

§ 7. The examples given above are particular cases of the following general theorem.

The function associated with a hypergeometric function is also a hypergeometric function.

I designate here by the name of hypergeometric function a function satisfying the differential equation of the second order

$$X_2 \frac{d^2y}{dx^2} + X_1 \frac{dy}{dx} + X_0 y = 0 \quad \text{............(33)},$$

X_2, X_1, X_0 being integral algebraical functions of X respectively of the second, first, and 0^{th} order in X. We shall not have to deal with hypergeometric functions of higher orders, as in the whole of the present paper we are concerned only with equations of the second order.

It is known that a hypergeometric function may be represented by Gauss' hypergeometric series

$$F(\alpha, \beta, \gamma, x) = 1 + \frac{\alpha.\beta}{1.\gamma} x + \frac{\alpha(\alpha+1)\beta(\beta+1)}{1.2\gamma(\gamma+1)} x^2 + \ldots (34).$$

We shall prove that if $y = F(\alpha, \beta, \gamma, x)$, the associated function η is also $= F(\alpha_1, \beta_1, \gamma_1, x)$, the elements α_1, β_1, γ_1 of the second series being connected with those of the first by means of some simple relations which we shall find.

The general equation of the hypergeometric series is

$$x(1-x)\frac{d^2y}{dx^2} + [\gamma - (\alpha+\beta+1)x]\frac{dy}{dx} - \alpha\beta y = 0 \ldots (35).$$

Thus we have

$$N : P : Q = x(x-1) : \gamma - (\alpha+\beta+1)x : -\alpha\beta$$

$$= \quad 1 \quad : \quad -\frac{x \log \Lambda}{dx} = -\Lambda L,$$

whence

$$\Lambda = cx^{-\gamma}(1-x)^{\gamma-\delta} \quad \text{..................(36)},$$

$$L = \frac{\alpha\beta}{c} x^{\gamma-1}(1-x)^{\delta-\gamma-1} \quad \text{............(37)},$$

where $\delta = \alpha + \beta + 1$, and

$$N_1 : P_1 : Q_1 = x(1-x) : 1 - \gamma + (\alpha+\beta-1)x : -\alpha\beta.$$

Thus the associated function η satisfies the differential equation

$$x(1-x)\frac{d^2y}{dx^2}+[\gamma_1-(\alpha_1+\beta_1+1)x]\frac{dy}{dx}-\alpha_1\beta_1 y=0 \ldots (38),$$

where the new elements α_1, β_1, γ_1 are connected with the old α, β, γ by the symmetric relations

$$\alpha+\alpha_1=0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (39),$$

$$\beta+\beta_1=0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (40),$$

$$\gamma+\gamma_1=1 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (41).$$

There is one exception to this rule, viz., when $\gamma=1$; for in this case $\gamma_1=0$, and the associated function is not a hypergeometric series, the symbol

$$F(\alpha, \beta, \gamma, x)$$

having no meaning for $\gamma=0$ (as also for $\gamma=$ any negative integer).

The hypergeometric series is finite, *i.e.* represents an integral algebraic function, if one of the interchangeable elements α, β is a negative integer; both y and η are integral algebraic functions if one of the elements α, β is a positive, the other a negative integer.

On the roots of associated functions, § 8.

§ 8. Theorem. *Associated functions have common multiple roots.* Suppose x_1 to be a root of y, repeated r times; then we have for this value of x,

$$y=0,\ \frac{dy}{dx}=0,\ \frac{d^2y}{dx^2}=0,\ \ldots,\ \frac{d^{r-1}y}{dx^{r-1}}=0,$$

but from (6) and (7) we find that in this case we have also

$$\eta=0,\ \frac{d\eta}{dx}=0,\ \frac{d^2\eta}{dx^2}=0,\ \ldots,\ \frac{d^{r-1}\eta}{dx^{r-1}}=0,$$

i.e., x_1 is also a root of η repeated r times.

Differential equation satisfied by the sum of two associated functions, each multiplied by any function of x, § 9.

§ 9. The sum of two associated functions, each multiplied by an arbitrary function of x, satisfies a differential equation of the same form as that satisfied by the functions themselves.

Let

$$u = Ry + P\eta \text{ ....(42)},$$

R and P being some functions of x. Differentiating, we have

$$\frac{du}{dx} = R\frac{dy}{dx} + P\frac{d\eta}{dx} + y\frac{dR}{dx} + \eta\frac{dP}{dx}$$

$$= \left(PL + \frac{dR}{dx}\right)y + \left(R\Lambda + \frac{dP}{dx}\right)\eta,$$

or shortly $$\frac{du}{dx} = R_1 y + P_1\eta \text{ ....(43)}.$$

In the same way we find

$$\frac{d^2u}{dx^2} = R_2 y + P_2\eta \text{ ....(44)};$$

eliminating from these three linear equations y and η, we shall have a linear relation between u, $\frac{du}{dx}$, and $\frac{d^2u}{dx^2}$ of the form

$$N\frac{d^2u}{dx^2} + P\frac{du}{dx} + Qu = 0 \text{ ....(45)},$$

Q. E. D.

Differential equation of the first order satisfied by the ratio of two associated functions, § 10.

§ 10. The ratio of the associated functions satisfying a differential equation of the first order and second degree (in the dependent variable) which we shall presently deduce. Let

$$u = \frac{y}{\eta}, \quad v = \frac{\eta}{y} \text{ ....(46)}.$$

We have

$$\frac{du}{dx} = \frac{1}{\eta}\frac{dy}{dx} - \frac{y}{\eta}\frac{1}{\eta}\frac{d\eta}{dx} = \Lambda - Lu^2 \text{ ....(47)},$$

$$\frac{dv}{dx} = \frac{1}{y}\frac{d\eta}{dx} - \frac{\eta}{y}\frac{1}{y}\frac{dy}{dx} = L - \Lambda v^2 \text{ ....(48)},$$

thus u and v satisfy the differential equations

$$\frac{du}{dx} + Lu^2 - \Lambda = 0 \text{ ....(49)},$$

$$\frac{dv}{dx} + \Lambda v^2 - L = 0 \text{ ....(50)}.$$

II.

Application of the theory of associated functions to a special case.

The complete elliptic integral K and Legendre's functions, § 11.

§ 11. It is known that Legendre's function $P_n(x)$ satisfies the differential equation of the second order,

$$(1-x^2)\frac{d^2P_n}{dx^2} - 2x\frac{dP_n}{dx} + n(n+1)z = 0 \text{ ......(51).}$$

If we transform this equation to the new independent variable k by means of the relation

$$x = 1 - 2k^2 \text{ ....(52),}$$

we find

$$k(1-k^2)\frac{d^2P_n}{dk^2} + (1-3k^2)\frac{dP_n}{dk} + 4n(n+1)kP_n = 0 \text{ ...(53).}$$

This gives for

$$n = -\tfrac{1}{2} \text{ ....(54),}$$

the equation

$$k(1-k^2)\frac{d^2P_{-\frac{1}{2}}}{dk^2} + (1-3k^2)\frac{dP_{-\frac{1}{2}}}{dk^2} - kP_{-\frac{1}{2}} = 0 \text{ ......(55).}$$

Again, from the theory of elliptic integrals we know that K satisfies the differential equation

$$k(1-k^2)\frac{d^2K}{dk^2} + (1-3k^2)\frac{dK}{dk} - kK = 0 \quad \text{...(56),}$$

whence we see that K may be considered as a special case of Legendre's function, viz.

$$K = P_{-\frac{1}{2}} \text{ ....(57).}$$

If we take the modular angle as the independent variable, the differential equation satisfied by K is

$$\frac{d^2K}{d\theta^2} + 2\cot 2\theta\frac{dK}{d\theta} - K = 0 \text{ ....(58).}$$

Again, if we take the angle ϕ whose cosine $= x$ as the independent variable in the equation satisfied by P_n, we find

$$\frac{d^2P_n}{d\phi^2} + \cot\phi\frac{dP_n}{d\phi} + n(n+1)P_n = 0 \text{ ....(59).}$$

These equations coincide for $n = -\frac{1}{2}$, and

$$\phi = 2\theta.$$

The complementary elliptic integral K' corresponds to Legendre's function of the second kind, usually denoted by Q_n. For the sake of symmetry, I shall denote this second solution by P_n'. Thus, the general solution of the equation (51) is

$$c_1 P_n + c_2 P_n' \quad \text{..................(60)},$$

as also the general solution of (56) is

$$c_1 K + c_2 K' \quad \text{....................(61)}.$$

The funtion associated with Legendre's function, § 12.

§ 12. We have already indicated in § 5 the function G associated with K. We shall now consider a general function R_n, associated with P_n, which shall become identical with G for $n = -\frac{1}{2}$.

Writing the differential equation satisfied by P_n in the form

$$\frac{d}{dx}\left\{(1 - x^2)\frac{dP_n}{dx}\right\} + n(n+1)P_n = 0 \quad \text{.......(62)},$$

we see that the associated function R_n is connected with P_n by the relation

$$(1 - x^2)\frac{dP_n}{dx} = cR_n \quad \text{................(63)},$$

c being an arbitrary constant. Thus we have

$$\frac{dP_n}{dx} = \frac{cR_n}{1 - x^2} \quad \text{....................(64)},$$

$$\frac{dR_n}{dx} = -\frac{n(n+1)}{c}P_n \quad \text{...........(65)},$$

and the new function R_n satisfies the differential equation

$$(1 - x^2)\frac{d^2R_n}{dx^2} + n(n+1)R_n = 0 \quad \text{........(66)};$$

for $x = 1 - 2k^2$, $n = -\frac{1}{2}$ we have

$$k(1 - k^2)\frac{d^2G}{dk^2} - (1 - k^2)\frac{dG}{dk} - kG = 0 \quad \text{.......(67)},$$

or also

$$\frac{d^2G}{d\theta^2} - 2\cot 2\theta\frac{dG}{d\theta} - G = 0 \quad \text{...........(68)},$$

in which form it appears very much like the equation

$$\frac{d^2K}{d\theta^2} - 2\cot 2\theta \frac{dK}{d\theta} - K = 0 \text{ ....(69)},$$

satisfied by K.

The complete solution of the equation (67) is

$$c_1 G + c_2 G' \text{ ....(70)},$$

G' being the complimentary function, *i.e.*

$$G' = E' - k^2 K' \text{ ....(71)}.$$

In the same way we may write as the general solution of the equation (66)

$$c_1 R_n + c_2 R'_n \text{ ....(72)},$$

R'_n being a new function which may be found from P'_n in the same way in which R_n is found from P_n.

Transformation to the independent variable $h = k^2$, § 13.

§ 13. Dr. Glaisher has shown that most of the formulæ relating to the elliptic integrals take a simpler form, if these quantities are considered as functions not of k itself, but of $k^2 = h$; the differential equations satisfied by K and G are in this case respectively

$$hh' \frac{d^2u}{dh^2} + (h' - h)\frac{du}{dh} - \tfrac{1}{4}u = 0 \text{ ....(73)},$$

$$u = c_1 K + c_2 K',$$

$$hh' \frac{d^2u}{dh^2} - \tfrac{1}{4}u = 0 \text{ ....(74)},$$

$$u = c_1 G + c_2 G',$$

where we have put $h' = 1 - h = k'^2$.

At the same time Legendre's function P_n, and the function R_n associated with it, satisfy the general equations

$$hh' \frac{d^2u}{dh^2} + (h' - h)\frac{du}{dh} + n(n+1)u = 0 \text{ ....(75)},$$

$$u = c_1 P_n + c_2 P'_n,$$

$$hh' \frac{d^2u}{dh^2} + n(n+1)u = 0 \text{ ....(76)},$$

$$u = c_1 R_n + c_2 R'_n.$$

In this form they appear as particular cases of the hypergeometric series, and we have

$$c_n P_n = F(n+1, -n, 1, h) \text{ ....}(77),$$

$$c_n R_n = F\{-(n+1), n, 0, h\} \text{............}(78),$$

this last symbol being, as it is written, without definite signification. Instead of it, we may write the associative relation existing between P_n and R_n.

The second elliptic integral E and the corresponding general function, U_n, § 14.

§ 14. The second elliptic integral E satisfies the differential equation

$$k(1-k^2)\frac{d^2E}{dk^2} + (1-k^2)\frac{dE}{dk} + kE = 0 \text{ ......}(79);$$

or, if h be taken as the independent variable,

$$hh'\frac{d^2E}{dh^2} + h'\frac{dE}{dh} + \tfrac{1}{4}E = 0 \text{ ....}(80),$$

In connection with it we consider a general function U_n satisfying the differential equation

$$hh'\frac{d^2U_n}{dh^2} + h'\frac{dU_n}{dh} + n^2U_n = 0 \text{ ....}(81).$$

This is identical with

$$c_n U_n = F(n, -n, 1, h) \text{...............}(82),$$

For $n = -\frac{1}{2}$, we have

$$cU_{-\frac{1}{2}} = E \text{.......................} (83).$$

The functions associated with U_n, § 15.

§ 15. We may easily find, by means of the processes indicated in part I, the functions V_n associated with U_n. We have, in the notation of the hypergeometric functions

$$c_n V_n = F(-n, n, 0, h) \text{ ....}(84),$$

but this has no direct meaning as $\gamma = 0$. The differential equation satisfied by V_n may be, however, directly found from this symbolic representation of V_n. It is

$$hh'\frac{d^2V_n}{dh^2} - h\frac{dV_n}{dh} + n^2V_n = 0 \text{............}(85),$$

and we have for $n = -\frac{1}{2}$

$$cV_{-\frac{1}{2}} = I \text{.......................}(86).$$

Complete solutions of the differential equations of the second order satisfied by U_n and V_n, § 16.

§ 16. If we change k into k', or h into h', we transform the unaccented letters K, G, E, I into the accented letters K', G', E', I'. By this change the equations satisfied by K and G and also by P_n and R_n are not altered, but the equations satisfied by E and I and also by U_n and V_n are converted into one another. Writing U_n', V_n' for functions of h' of the same form as U_n, V_n are of h, we have therefore for the second solution of the equation satisfied by U_n, the function V_n', and for the second solution of the equation satisfied by V_n, the function U_n'. Thus we may write

$$hh'\frac{d^2u}{dh^2}+h'\frac{du}{dh}+n^2u=0 \qquad (87),$$

$$u=c_1U_n+c_2V_n',$$

$$hh'\frac{d^2u}{dh^2}-h\frac{du}{dh}+n^2u=0 \qquad (88),$$

$$u=c_1V_n+c_2U_n'.$$

Transformation to the independent variable x, § 17.

§ 17. If we take in U_n and V_n as the independent variable the same quantity x, in which P_n is usually expressed, we shall have the following differential equations satisfied by U_n and V_n respectively,

$$(1-x^2)\frac{d^2u}{dx^2}+(1+x)\frac{du}{dx}+n^2u=0 \qquad (89),$$

$$u=c_1U_n+c_2V_n',$$

$$(1-x^2)\frac{d^2u}{dx^2}-(1-x)\frac{du}{dx}+n^2u=0 \qquad (90),$$

$$u=c_1V_n+c_2U_n'.$$

The associative relation existing between U_n, V_n is

$$\frac{dU_n}{dx}=\frac{cV_n}{1-x} \qquad (91),$$

$$\frac{dV_n}{dx}=\frac{-n^2U_n}{c(1+x)} \qquad (92),$$

c being an arbitrary constant.

Choice of the arbitrary constants, § 18.

§ 18. The constant c entering in the associative relations between P_n and R_n, and between U_n and V_n may be chosen arbitrarily, with the only restrictions that for $n = -\frac{1}{2}$, when P_n, R_n, U_n, V_n become K, G, E, I respectively, we must have

$$G = E - k'K, \quad I = E - K,$$

But we may find for them such values that the same relations will hold good for any value of n, *i.e.* so that we shall have, independently of n,

$$R = U - k'P, \quad V = U - P.$$

In order to have this, we must take c in the formulæ (64), (65) equal to $2n$, and in the formulæ (91), (92), $c = n$ (as will be seen from § 20). We have then for these associative relations

$$\frac{dP_n}{dx} = \frac{2nR_n}{1-x^2} \quad \text{.................. (93)},$$

$$\frac{dR_n}{dx} = -\tfrac{1}{2}(n+1)P_n \quad \text{........... (94)},$$

$$\frac{dU_n}{dx} = \frac{nV_n}{1-x} \quad \text{.................... (95)},$$

$$\frac{dV_n}{dx} = -\frac{nU_n}{1+x} \quad \text{.................... (96)}.$$

List of formulae involving P_n, R_n, U_n, V_n, § 19.

§ 19. I give in this section a list of the formulæ connecting P_n, R_n, U_n, V_n, expressed in four variables x, k, h, θ, where

$$x = 1 - 2k^2 = 1 - 2h = \cos 2\theta \quad \text{...........(97)}.$$

Some of these formulæ have been found in the preceding sections, some others may be easily deduced from them by simple operations:

(i)

$$(1-x^2)\frac{d^2u}{dx^2} - 2x\frac{du}{dx} + n(n+1)u = 0,$$

$$u = c_1P_n + c_2P_n',$$

$$(1-x^2)\frac{d^2u}{dx^2} \qquad\qquad + n(n+1)u = 0,$$

$$u = c_1R_n + c_2R_n',$$

$$(1-x^2)\frac{d^2u}{dx^2}+(1+x)\frac{du}{dx}+n^2u=0,$$

$$u=c_1U_n+c_2V_n',$$

$$(1-x^2)\frac{d^2u}{dx^2}-(1-x)\frac{du}{dx}+n^2u=0,$$

$$u=c_1V_n+c_2U_n',$$

$$\frac{dP_n}{dx}=\quad\frac{2nR_n}{1-x^2},$$

$$\frac{dR_n}{dx}=-\tfrac{1}{2}(n+1)P_n,$$

$$\frac{dU_n}{dx}=\quad\frac{nV_n}{1-x},$$

$$\frac{dV_n}{dx}=-\frac{nU_n}{1-x},$$

(ii).

$$h(1-h)\frac{d^2u}{dh^2}+(1-2h)\frac{du}{dh}+n(n+1)u=0,$$

$$u=c_1P_n+c_2P_n',$$

$$h(1-h)\frac{d^2u}{dh^2}+\qquad\qquad+n(n+1)u=0,$$

$$u=c_1R_n+c_2R_n',$$

$$h(1-h)\frac{d^2u}{dh^2}+(1-\quad h)\frac{du}{dh}+n^2u=0,$$

$$u=c_1U_n+c_2V_n',$$

$$h(1-h)\frac{d^2u}{dh^2}+\qquad h\frac{du}{dh}+n^2n=0,$$

$$u=c_1V_n+c_2U_n',$$

$$P_n=F(n+1,-n,1,h),$$

$$U_n=F(\quad n\quad,-n,1,h),$$

$$\frac{dP_n}{dh} = -\frac{nR_n}{h(1-h)},$$

$$\frac{dR_n}{dh} = \quad (n+1)\,P_n,$$

$$\frac{dU_n}{dh} = -\frac{n}{h}\,V_n,$$

$$\frac{dV_n}{dh} = \quad \frac{n}{1-h}\,U_n.$$

III.

$$\frac{d^2u}{d\theta^2} + 2\cot 2\theta\,\frac{du}{d\theta} + 4n(n+1)\,u = 0,$$

$$u = c_1 P_n + c_2 P_n',$$

$$\frac{d^2u}{d\theta^2} - 2\cot 2\theta\,\frac{du}{d\theta} + 4n(n+1)\,u = 0,$$

$$u = c_1 R_n + c_2 R_n',$$

$$\frac{d^2u}{d\theta^2} + 2\sec 2\theta\,\frac{du}{d\theta} + 4n^2\,u = 0,$$

$$u = c_1 U_n + c_2 V_n',$$

$$\frac{d^2u}{d\theta^2} - 2\sec 2\theta\,\frac{du}{d\theta} + 4n^2 u = 0,$$

$$u = c_1 V_n + c_2 U_n',$$

$$\frac{dP_n}{d\theta} = -\frac{2n\,R_n}{\sin\theta\cos\theta},$$

$$\frac{dR_n}{d\theta} = \quad 2(n+1)\sin\theta\cos\theta P_n,$$

$$\frac{dU_n}{d\theta} = -\,2n\tan\theta\,V_n,$$

$$\frac{dV_n}{d\theta} = \quad 2n\cot\theta\,U_n,$$

$$k(1-k^2)\frac{d^2u}{dk^2} + (1-3k^2)\frac{du}{dk} + 4n(n+1)\,ku = 0,$$

$$u = c_1 P_n + c_2 P_n',$$

$$k(1-k^2)\frac{d^2u}{dk^2}-(1-k^2)\frac{du}{dk}+4n(n+1)ku=0,$$

$$u=c_1R_n+c_2R_n',$$

$$k(1-k^2)\frac{d^2u}{dk^2}+(1-k^2)\frac{du}{dk}+4n^2ku=0,$$

$$u=c_1U_n+c_2V_n',$$

$$k(1-k^2)\frac{d^2u}{dk^2}-(1+k^2)\frac{du}{dk}+4n^2ku=0,$$

$$u=c_1V_n+c_2U_n'.$$

I adjoin also the formulæ transformed to the variable $\phi=2\theta$, *i.e.* the angle ϕ for which $\cos\phi=x$. The system (iii) takes a somewhat simpler form in this case, as the numerical coefficients disappear altogether in the differential equations. Thus we have

$$\frac{d^2u}{d\phi^2}+\cot\phi\frac{du}{d\phi}+n(n+1)u=0,$$

$$u=c_1P_n+c_2P_n',$$

$$\frac{d^2u}{d\phi^2}-\cot\phi\frac{du}{d\phi}+n(n+1)u=0,$$

$$u=c_1R_n+c_2R_n',$$

$$\frac{d^2u}{d\phi^2}+\operatorname{cec}\phi\frac{du}{d\phi}+n^2u=0,$$

$$u=c_1U_n+c_2V_n',$$

$$\frac{d^2u}{d\phi^2}-\operatorname{cec}\phi\frac{du}{d\phi}+n^2u=0,$$

$$u=c_1V_n+c_2U_n',$$

$$\frac{dP_n}{d\phi}=-2n\operatorname{cec}\phi R_n,$$

$$\frac{dR_n}{d\phi}=\quad(n+1)\sin\phi P_n,$$

$$\frac{dU_n}{d\phi}=-\frac{n\sin\phi}{1+\cos\phi}V_n,$$

$$\frac{dV_n}{d\phi}=\quad\frac{n\cos\phi}{1-\cos\phi}U_n.$$

Relations between three functions of the group P_n, R_n, U_n, V_n, § 20.

§ 20. The four functions P_n, R_n, U_n, V_n constitute a group of quantities most intimately connected with one another. Any of these quantities may be expressed by means of any two others linearly. It is known from Gauss's theory of the hypergeometric series that we have

$$F(\alpha+1, \beta+1, \gamma+1, x)$$

$$= \frac{\gamma}{\beta}\frac{1}{x}[F(\alpha+1, \beta, \gamma, x) - F(\alpha, \beta, \gamma, x)] \text{......(98)},$$

$$\frac{d}{dx}F(\alpha, \beta, \gamma, x) = \frac{\alpha\beta}{\gamma}F(\alpha+1, \beta+1, \gamma+1, x) \text{...(99)},$$

whence we find

$$\frac{d}{dx}F(\alpha, \beta, \gamma, x) = \frac{\alpha}{x}[F(\alpha+1, \beta, \gamma, x) - F(\alpha, \beta, \gamma, x)] \text{ (100)}.$$

Putting here

$$\alpha = n,\ \beta = -n,\ \gamma = 1,\ x = h \text{ ....(101)},$$

we have

$$\frac{d}{dh}F(n, -n, 1, h) = \frac{n}{h}\{F(n+1, -n, 1, h) - F(n, -n, 1, h)\},$$

or else

$$h\frac{dU_n}{dh} + nU_n = nP_n \text{ ....(102)},$$

or again, in virtue of § 19,

$$P_n = U_n - V_n,$$

i.e. independently of n,

$$P = U - V \text{.....................(103)}.$$

In the same way we find

$$U = R + (1-h)P \text{ ....(104)},$$

whence we may deduce also the expression connecting P, R, V and R, U, V. We have thus the following set of finite relations between P, R, U, V:

$$R - Uh - Vh' = 0 \text{ ....(105)},$$

$$P - U + V = 0 \text{ ....(106)},$$

$$Ph - R + V = 0 \text{ ....(107)},$$

$$Ph' + R - U = 0 \text{ ....(108)}.$$

Formulæ involving the derivatives of P_n, R_n, U_n, V_n, and also formulæ for P_{n+1}, R_{n+1}, U_{n+1}, V_{n+1}, &c., § 21.

§ 21. Substituting for P, R, U, V, their values expressed by means of the derivatives of the corresponding associated function, we may find a group of formulæ giving the value of any derivative expressed by means of any two functions of the group P_n, R_n, U_n, V_n, $\frac{dP_n}{dh}$, $\frac{dR_n}{dh}$, $\frac{dU_n}{dh}$, $\frac{dV_n}{dh}$. Thus, *e.g.*, we have

$$h\frac{dU_n}{dh} = n\,(hP_n - R_n), \text{\&c.}$$

The number of such formulæ is 28, viz. (i) 8 expressing any quantities P, R, U, V, by means of a second quantity together with its derivative, *e.g.*

$$U_n = h'P_n - \frac{hh'}{n}\,\frac{dP_n}{dh},$$

(ii) 4 expressions for the derivative of any of these quantities by means of two functions, *e.g.*

$$\frac{dP_n}{dh} = -\frac{n}{hh'}\,(hU + h'V),$$

(iii) 8 expressions for the derivative of any of the two quantities by means of any other quantity together with its derivative, as

$$\frac{dU_n}{dh} = nP_n + h'\,\frac{dP_n}{dh},$$

(iv) 4 expressions for any of the quantities by means of the derivatives of any two other quantities, *e.g.*

$$P = -\frac{h}{n}\,\frac{dU_n}{dh} + \frac{h'}{n}\,\frac{dV_n}{dh},$$

and, lastly, (v) 4 expressions for the derivative of any quantity by means of the derivatives of any two other quantities, as *e.g.*

$$\frac{dP_n}{dh} = \frac{dU_n}{dh} - \frac{dV_n}{dh}.$$

Again, there exists a great number of relations connecting three consecutive quantities of the same kind, such as

$$P_{n+1}, P_n, P_{n-1}, R_{n+1}, R_n, R_{n-1}, U_{n+1}, U_n, U_{n-1}, V_{n+1}, V_n, V_{n-1},$$

or expressing any of these 12 quantities, or of their derivatives, by means of any two other quantities of the complete series of 24 quantities, the 12 just written and their 12 derivatives. The formulæ are analogous to the known relations between three consecutive spherical harmonics and other similar formulæ known in the theory of these functions.

The quantities W and H, § 22.

§ 22. I have mentioned in § 5 a quantity W, introduced by Dr. Glaisher, who has found that it bears a close analogy to K, and the quantity H, which I introduced in the paper quoted above, as being connected with W by a relation analogous to that connecting G and K. These two quantities appear now from the general point of view taken in this paper as the first elements of an infinite series of similar functions of the same kind. In fact, the quantity W satisfies the differential equation

$$h(1-h)\frac{d^2u}{dh^2}(1-2h)\frac{du}{dh}+\tfrac{3}{4}u=0 \quad \text{......(109)};$$

whence we at once conclude that

$$W=cP_{-\frac{3}{2}} \quad \text{.....................(110)}.$$

In the same way we have for the differential equation satisfied by H,

$$h(1-h)\frac{d^2u}{dh^2}+\tfrac{3}{4}u=0 \quad \text{...............(111)},$$

whence $$H=cR_{-\frac{3}{2}} \quad \text{.......................(112)},$$

or also we may write

$$W=cP_{+\frac{1}{2}},\ H=cR_{+\frac{1}{2}} \quad \text{.............(113)}.$$

Note. As $n(n+1)$ is left unchanged by the change of n into $-(n+1)$, we have generally $P_i=P_{-(i+1)}$, and therefore also $R_i=R_{-(i+1)}$, if P, R denote generally any solution of the differential equations above.

Thus, W and H are, together with K and G, the first terms of the infinite series of functions

$$P_{-\frac{1}{2}},\ P_{-\frac{3}{2}},\ P_{-\frac{5}{2}}, \ldots, P_{-\frac{1}{2}(2i+1)}, \ldots,$$

$$R_{-\frac{1}{2}},\ R_{-\frac{3}{2}},\ R_{-\frac{5}{2}}, \ldots, R_{-\frac{1}{2}(2i+1)}, \ldots,$$

or else they are the two middle terms of the series

$$\ldots P_{-\frac{3}{2}},\ P_{-\frac{1}{2}},\ P_{\frac{1}{2}},\ P_{\frac{3}{2}}, \ldots,$$

$$\ldots R_{-\frac{3}{2}},\ R_{-\frac{1}{2}},\ R_{\frac{1}{2}},\ R_{\frac{3}{2}}, \ldots.$$

Supposing K_0, G_0 to denote K, G respectively, we shall write also

$$2W = K_1,\ H = G_1,$$

and generally

$$c_i P_{-\frac{1}{2}(2i+1)} = K_i,\ c_i R_{-\frac{1}{2}(2i+1)} = G_i \ \ldots\ldots\ldots(114),$$

and we have thus the two infinite series of functions

$$K_0,\ K_1,\ K_2, \ldots,$$

$$G_0,\ G_1,\ G_2, \ldots,$$

of which the first two terms are known in both cases. Now as there exists a recurrent relation between any three P's or any three R's, there exists a similar relation between any three K's and any three G's; and knowing two of them we may construct any other. Thus, we may consider the whole series as known.

The quantities K_i, G_i, § 23.

§ 23. The quantities K_i, G_i, introduced in the preceding section, constitute a class of functions which it is convenient to consider separately for themselves. The differential equations satisfied by them are

$$hh' \frac{d^2 K_i}{dh^2} + (h - h') \frac{dK_i}{dh} + (i^2 - \tfrac{1}{4}) K_i = 0 \ \ldots.(115),$$

$$hh' \frac{d^2 G_i}{dh^2} \qquad\qquad + (i^2 - \tfrac{1}{4}) G_i = 0 \ \ldots.(116).$$

We shall denote by K_i' and G_i' respectively the same functions of h' as K_i, G_i are of h. Thus the complete solutions of the equations just written are

$$c_1 K_i + c_2 K_i',$$
$$c_1 G_i + c_2 G_i'.$$

The differential associative relations between K_i and G_i are

$$\frac{dK_i}{dh} = \frac{(2i+1)}{hh'} G_i \text{ ....(117)},$$

$$\frac{dG_i}{dh} = -(2i-1) K_i \text{ ....(118)},$$

and, lastly, we have

$$c_i K_i = F(\tfrac{1}{2}+i, \tfrac{1}{2}-i, 1, h) \text{ ....(119)}.$$

The quantities E_i, I_i, § 24.

§ 24. In the same way as we have defined the functions K_i, G_i as particular cases of P_n, R_n for $2n =$ odd integer, we define two new functions E_i, I_i by the relations

$$c_i E_i = U_{-\frac{1}{2}(2i+1)}, \; c_i I_i = V_{-\frac{1}{2}(2i+1)} \text{ ....(120)}.$$

We shall thus have an infinite series of functions

$$\ldots E_{-1}, E_0, E_1, \ldots,$$
$$\ldots I_{-1}, I_0, I_1, \ldots,$$

of which $E_0 = E$, $I_0 = I_1$ are already known, and the others may be found from the known values of K_i, G_i, by means of the equations of § 20. The differential equations satisfied by these functions are

$$hh' \frac{d^2 E_i}{dh^2} + h' \frac{dE_i}{dh} + [i(i+1) + \tfrac{1}{4}] E_i = 0 \text{ ...(121)},$$

$$hh' \frac{d^2 I_i}{dh^2} - h \frac{dI_i}{dh} + [i(i+1) + \tfrac{1}{4}] I_i = 0 \text{ ...(122)}.$$

The differential associative relations between E_i and I_i are

$$\frac{dE_i}{dh} = \frac{1+2i}{h} I_i \text{ ....(123)},$$

$$\frac{dI_i}{dh} = \frac{1-2i}{h'} E_i \text{ ....(124)},$$

and, lastly,

$$c_i E_i = F\{\tfrac{1}{2}(2i+1), -\tfrac{1}{2}(2i+1), 1, h\} \text{ ...(125)}.$$

The complete solutions of the above differential equations are easily found to be

$$c_1 E_i + c_2 I_i' \quad\ldots\ldots\ldots\ldots(126),$$

$$c_1 I_i + c_2 E_i' \quad\ldots\ldots\ldots\ldots(127),$$

E_i', I_i' denoting the same function of h' as E_i, I_i are of h.

Development of P_n, R_n, U_n, V_n in series, § 25.

§ 25. The hypergeometric expression of P_n, U_n, gives at once the development of the two functions in ascending powers of h. Thus

$$c_n P_n = 1 - \frac{(n+1)\,n}{1^2} h + \frac{(n+2)(n+1)\,n\,(n-1)}{1^2.2^2} h^2$$
$$- \frac{(n+3)(n+2)(n+1)\,n\,(n-1)(n-2)}{1^2.2^2.3^2} h^3 + \ldots (128),$$

$$c_n U_n = 1 - \frac{n^2}{1^2} h + \frac{n^2(n^2-1^2)}{1^2.2^2} h^2 - \frac{n^2(n^2-1^2)(n^2-2^2)}{1^2.2^2.3^2} h^3 + \ldots (129);$$

c_n denotes here an arbitrary constant.

It is easy to prove that U_n contains $1-h$ as a factor for every integer value of n, excepting $n=0$ (as $U_0=1$). In the same way we may easily show that V_n contains h, and R_n $h(1-h)$, as factors.

For if we put $h-1$ in the series of U_n, we have

$$(c_n U_n)_1 = 1 - \frac{n^2}{1^2} + \frac{n^2(n^2-1^2)}{1^2.2^2} - \frac{n^2(n^2-1^2)(n^2-2^2)}{1^2.2^2.3^2} + \ldots = 0$$
$$\ldots\ldots\ldots\ldots(130).$$

Now from the differential relations existing between P_n and R_n, U_n and V_n, we see that R_n vanishes for $h=0$ and $h'=0$, and V_n vanishes for $h=1$.

We shall therefore consider three new functions u_n, v_n, r_n defined by the equations

$$R_n = c_n h h' r_{n-1},\quad U_n = c_n h' u_{n-1},\quad V_n = c_n h v_{n-1} \ldots (131).$$

Remark. Here, as everywhere, we denote generally by c, c_n, c_i, &c. any arbitrary constants which may be different in different formulæ. Thus in the three equations (131) c_n may be different in each case.

We shall write also for symmetry

$$P_n = c_n p_n \ldots\ldots\ldots\ldots(132).$$

Thus the differential equations satisfied by p_n, r_n, u_n, v_n are

$$hh'\frac{d^2p_n}{dh^2}+(h'-h)\frac{dp_n}{dh}+n(n+1)p_n=0 \ldots(133),$$

$$hh'\frac{d^2r_n}{dh^2}+2(h'-h)\frac{dr_n}{dh}+n(n+3)r_n=0 \ldots(134),$$

$$hh'\frac{d^2u_n}{dh^2}+(h'-2h)\frac{du_n}{dh}+n(n+2)u_n=0 \ldots(135),$$

$$hh'\frac{d^2v_n}{dh^2}-(h-2h')\frac{dv_n}{dh}+n(n+2)v_n=0 \ldots(136).$$

All these functions are special cases of the hypergeometric series, viz.

$$c_np_n=F(n+1,\ -n,\ 1,\ h) \ldots\ldots\ldots\ldots(137),$$

$$c_nr_n=F(n+3,\ -n,\ 2,\ h) \ldots\ldots\ldots\ldots(138),$$

$$c_nu_n=F(n+2,\ -n,\ 1,\ h) \ldots\ldots\ldots\ldots(139),$$

$$c_nv_n=F(n+2,\ -n,\ 2,\ h) \ldots\ldots\ldots\ldots(130).$$

Thus we write at once the following series for p_n, r_{n-1}, u_{n-1}, v_{n-1}:

$$c_np_n=1-\frac{n(n+1)}{1^2}h+\frac{n(n+2)}{1^2}\frac{n^2-1^2}{2^2}h^2$$
$$-\frac{n(n+3)}{1^2}\frac{n^2-1^2}{2^2}\frac{n^2-2^2}{3^2}h^3+\ldots(141),$$

$$c_{n+1}r_{n-1}=1-\frac{(n-1)(n+2)}{2}h+\frac{(n-1)(n+3)}{3^4}\frac{n^2-2^2}{2^2}h^2$$
$$-\frac{(n-1)(n+4)}{4}\frac{n^2-2^2}{2^2}\frac{n^2-3^2}{3^2}h^3+\ldots(142),$$

$$c_{n+1}u_{n-1}=1-\frac{n^2-1^2}{1^2}h+\frac{n^2-1^2}{1^2}\frac{n^2-2^2}{2^2}h^2$$
$$-\frac{n^2-1^2}{1^2}\frac{n^2-2^2}{2^2}\frac{n^2-3^2}{3^2}h^3+\ldots(143),$$

$$c_{n+1}v_{n-1}=1-\frac{n^2-1^2}{2}h+\frac{n^2-1^2}{3}\frac{n^2-2^2}{2^2}h^2$$
$$-\frac{n^2-1^2}{4}\frac{n^2-2^2}{2^2}\frac{n^2-3^2}{3^2}h^3+\ldots(144).$$

Examples of the four functions, § 26.

§ 26. Putting in these series $n=0, 1, 2$, &c. we find the following expressions for the first terms of the infinite series of functions p_n, r_n, u_n, v_n, the first of them being the spherical harmonic function.

(i).

$$c_0p_0 = 1,$$
$$c_1p_1 = 1 - 2h,$$
$$c_2p_2 = 1 - 6h + 6h^2,$$
$$c_3p_3 = 1 - 12h + 60h^2 - 252h^3,$$
....

(ii).

$$c_0r_0 = 1,$$
$$c_1r_1 = 1 - 2h,$$
$$c_2r_2 = 1 - 5h + 5h^2,$$
$$c_3r_3 = 1 - 9h + 21h^2 - 14h^3,$$
....

(iii).

$$c_0u_0 = 1,$$
$$c_1u_1 = 1 - 3h,$$
$$c_2u_2 = 1 - 8h + 10h^2,$$
$$c_3u_3 = 1 - 15h + 45h^2 - 35h^3,$$
....

(iv).*

$$c_0v_0 = 1,$$
$$c_1v_1 = 2 - 3h,$$
$$c_2v_2 = 3 - 12h + 10h^2,$$
$$c_3v_3 = 4 - 30h + 60h^2 - 35h^3,$$
....

* In order to avoid fractions in the expression of v I have taken the constant equal to nc_n instead of c_n.

Choice of the arbitrary constants, § 27.

§ 27. The constants c_n may be chosen arbitrarily. But we shall choose them so that P_n shall be identical with Legendre's function usually designated by this letter. Therefore c_n in (77) must be put $=1$, as also in the series (128), which follows from (77). Then also we shall have $c_n = 1$ in the corresponding expressions of R_n, U_n, V_n, the constants in p_n, r_n, u_n, v_n remaining still arbitrary. Again we shall choose the constants c_i in the expressions (114) so that for $i=0$ they shall give the known functions K, G, E as also in (120), so that they shall give for $i=0$ the known functions E, I. We may put for any n or i

$$c_i = \frac{2}{\pi} \qquad \text{.......................(145).}$$

Series for K_i, G_i, E_i, I_i, § 28.

§ 28. Putting in the series (128), (129) $n = -\frac{1}{2}(2i+1)$, we shall have the corresponding series for K_i, G_i; we may find also series for E_i, I_i from the relations (105)–(108), or also from the series (143), (144) for u_n, v_n.

We find, by the first method,

$$\frac{2K_i}{\pi} = 1 - \frac{i^2-\frac{1}{4}}{1^2}h + \frac{i^2-\frac{1}{4}}{1^2}\,\frac{i^2-\frac{9}{4}}{2^2}h^2 - \frac{i^2-\frac{1}{4}}{1^2}\,\frac{i^2-\frac{9}{4}}{2^2}\,\frac{i^2-\frac{25}{4}}{3^2}h^3 + \text{......(146)},$$

$$\frac{2G_i}{\pi} = -\frac{2i-1}{2}h\left\{1 - \frac{i^2-\frac{1}{4}}{2}h + \frac{i^2-\frac{1}{4}}{3}\,\frac{i^2-\frac{9}{4}}{2^2}h^2 - \frac{i^2-\frac{1}{4}}{4}\,\frac{i^2-\frac{9}{4}}{2^2}\,\frac{i^2-\frac{25}{4}}{3^2}h^3 + \dots\right\}\text{...(147)},$$

$$\frac{2E_i}{\pi} = 1 - \frac{i(i+1)+\frac{1}{4}}{1^2}h + \frac{i(i+1)+\frac{1}{4}}{1^2}\,\frac{i(i+1)-\frac{1}{4}(1.3)}{2^2}h^2 - \frac{i(i+1)+\frac{1}{4}}{1^2}\,\frac{i(i+1)-\frac{1}{4}(1.3)}{2^2}\,\frac{i(i+1)-\frac{1}{4}(3.5)}{3^2}h^3 + \text{...(148)},$$

$$\frac{2I_i}{\pi} = -\frac{2i+1}{2}h\left\{1 - \frac{i(i+1)-\frac{1}{4}(1.3)}{2}h + \frac{i(i+1)-\frac{1}{4}(1.3)}{3}\,\frac{i(i+1)-\frac{1}{4}(3.5)}{2^2}h^2 - \frac{i(i+1)-\frac{1}{4}(1.3)}{4}\,\frac{i(i+1)-\frac{1}{4}(3.5)}{2^2}\,\frac{i(i+1)-\frac{1}{4}(5.7)}{3^2}h^3 + \dots\right\}\text{(149).}$$

Thus *e.g.* for $i=0$, we find the known series

$$\frac{2K}{\pi}= \quad 1+\frac{1}{2^2}k^2+\frac{1^2.3^2}{2^2.4^2}k^4+\frac{1^2.3^2.5^2}{2^2.4^2.6^2}k^6+\ldots,$$

$$\frac{2G}{\pi}= \quad \tfrac{1}{2}k^2\left[1+\frac{1}{4^2}2k^2+\frac{1^2.3^2}{4^2.6^2}3k^4+\frac{1^2.3^2.5^2}{4^2.6^2.8^2}4k^6+\ldots\right],$$

$$\frac{2E}{\pi}= \quad 1-\frac{1}{2^2}k^2-\frac{1^2.3}{2^2.4^2}k^4-\frac{1^2.3^2.5}{2^2.4^2.6^2}k^6-\ldots,$$

$$\frac{2I}{\pi}=-\tfrac{1}{2}k'^2\left[1+\frac{1^2.3}{2^2.2}k^2+\frac{1^2.3^2.5}{2^2.4^2.3}k^4+\frac{1^2.3^2.5^2.7}{2^2.4^2.6^2.4}k^6+\ldots\right].$$

The series for u_n, v_n give also

$$\frac{2G_i}{\pi}=-\frac{2i-1}{2}h(1-h)\left(1-\frac{i^2-\frac{9}{4}}{2}h\right.$$

$$\left.+\frac{(i^2-\frac{9}{4})(i^2-\frac{25}{4})}{3.2^2}h^2-\frac{(i^2-\frac{9}{4})(i^2-\frac{25}{4})(i^2-\frac{49}{4})}{4.2^2.3^2}h^3+\ldots\right)(150)$$

$$\frac{2E_i}{\pi}= \quad (1-h)\left(1-\frac{i(i+1)-\frac{1}{4}(1.3)}{1^2}h\right.$$

$$\left.+\frac{i(i+1)-\frac{1}{4}(1.3)}{1^2}\,\frac{i(i+1)-\frac{1}{4}(3.5)}{2^2}h^2\ldots\right)\ldots(151),$$

whence we derive for $i=0$ the new series

$$\frac{2G}{\pi}=\tfrac{1}{2}k^2(1-k^2)\left(1+\frac{3^2}{4^2}2k^2+\frac{3^2.5^2}{4^2.6^2}3k^4+\frac{3^2.5^2.7^2}{4^2.6^2.8^2}4k^6+\ldots\right)(152),$$

$$\frac{2E}{\pi}=(1-k^2)\left(1+\frac{3}{2^2}k^2+\frac{3^2.5}{2^2.4^2}k^4+\frac{3^2.5^2.7}{2^2.4^2.6^2}k^6+\ldots\right)\ \ldots\ldots(153).$$

Transformation to the variable x, § 29

§ 29. The functions P_n, R_n, U_n, V_n and K_i, G_i, E_i, I_i may be also developed in elegant series proceeding in ascending powers of the variable $x=1-2h=h'-h$. The development of P_n is known from the theory of spherical harmonic functions. That of R_n may be found by simple integration of the series; and from the series for P_n and R_n those for U_n, V_n may be found by simple algebraical processes. Again by changing n into $-\frac{1}{2}(2i+1)$ we shall find series for K_i, G_i, E_i and I_i. These series are for P_n symbolically,

$$P_{2m}\ =F(-m,\ m+1,\ \tfrac{1}{2},\ x^2)\ \ldots\ldots\ldots(154),$$

$$P_{2m+1}=xF(-m,\ m+\tfrac{3}{2},\ \tfrac{1}{2},\ x^2)\ \ldots\ldots(155),$$

or in developed form, writing $n=2m$ or $n=2m+1$ in these formulæ,

$$P_n = 1 - \frac{n(n+1)}{2!}x^2 + \frac{(n-2)n(n+1)(n+3)}{4!}x^4 - \dots \text{ for } n \text{ even} \quad \dots\dots(156),$$

$$P_n = x\left(1 - \frac{(n+2)(n-1)}{3!}x^2 + \frac{(n+4)(n+2)(n-1)(n-3)}{5!}x^4 - \dots\right) \text{ for } n \text{ odd} \quad \dots\dots(157).$$

But we have

$$R_n = -\tfrac{1}{2}(n+1)\int P_n dx + C.$$

Thus taking in account also the second associative formula existing between P_n and R_n which determines the value of the constant, we find

I.

$$R_n = -\tfrac{1}{2}(n+1)\,x\left(1 - \frac{n(n+1)}{3!}x^2 + \frac{(n-2)n(n+1)(n+3)}{5!}x^4 - \dots\right) \text{ for } n \text{ even} \quad \dots\dots(158),$$

$$R_n = -\tfrac{1}{2}(n+1)\left(1 + \frac{x^2}{2!} - \frac{(n-1)(n+2)}{4!}x^4 + \dots\right) n \text{ odd} \quad (159),$$

and, for any n,

$$U_n = 1 - \frac{n-1}{1!}x - \frac{(n-1)(n-2)}{2!}x^2 - \frac{(n-1)(n-3)(n-2)}{3!}x^3 - \dots \quad \dots\dots(160),$$

$$V_n = 1 + \frac{n-1}{1!}x - \frac{(n-1)(n+2)}{2!}x^2 + \frac{(n-1)(n-3)(n+2)}{3!}x^3 - \dots\dots\dots\dots(161).$$

II.

$$\frac{2K_i}{\pi}=1-\frac{i^2-\frac{1}{4}}{2!}x^2+\frac{(i^2-\frac{1}{4})(i^2-\frac{25}{4})}{4!}x^4$$
$$-\frac{(i^2-\frac{1}{4})(i^2-\frac{25}{4})(i^2-\frac{81}{4})}{6!}x^6+\dots\dots\dots\dots \quad (162),$$

or else

$$\frac{2K_i}{\pi}=1-\frac{4i^2-1^2}{2^2.2!}x^2+\frac{(4i^2-1^2)(4i^2-5^2)}{2^4.4!}x^4$$
$$-\frac{(4i^2-1^2)(4i^2-5^2)(4i^2-9^2)}{2^6.6!}x^6\dots\dots(163),$$

where the numerators of the coefficients contain the product of the difference of $(2i)^2$, and the powers of the numbers of the form $4p+1$. For $i=0$ we have

$$\frac{2K}{\pi}=1+\frac{1^2}{2^2.2!}x^2+\frac{1^2.5^2}{2^4.4!}x^4+\frac{1^2.5^2.9^2}{2^6.6!}x^6+\dots\dots\dots\dots(164).$$

Again we have for $\frac{2G_i}{\pi}$ the series

$$\frac{2G_i}{\pi}=\tfrac{1}{2}(2i-1)x\left(1-\frac{i^2-\frac{1}{4}}{3!}x^2+\frac{(i^2-\frac{1}{4})(i^2-\frac{25}{4})}{5!}x^4\right.$$
$$\left.-\frac{(i^2-\frac{1}{4})(i^2-\frac{25}{4})(i^2-\frac{81}{4})}{7!}x^6+\dots\right)\dots\dots(165).$$

or else

$$\frac{2G_i}{\pi}=\frac{2i-1}{2}x\left(1-\frac{4i^2-1^2}{2^2.3!}x^3+\frac{(4i^2-1^2)(4i^2-5^2)}{2^4.5!}x^4\right.$$
$$\left.-\frac{(4i^2-1^2)(4i^2-5^2)(4i^2-9^2)}{2^6.7!}x^6+\dots\right)\dots(166),$$

e.g. for $i=0$

$$\frac{2G}{\pi}=-\tfrac{1}{2}x\left(1+\frac{1}{2^2.3!}x^2+\frac{1^2.5^2}{2^4.5!}x^4+\frac{1^2.5^2.9}{2^6.7!}x^6+\dots\right)\dots(167).$$

In the same way we have for E_i and I_i

$$\frac{2E_i}{\pi}=1+\frac{2i+3}{2}x+\frac{4i^2-3^2}{2^2}\frac{x^2}{2!}+\frac{4i^2-3^2}{2^2}\frac{2i+7}{2}\frac{x^3}{3!}+\dots(168),$$

$$\frac{2I_i}{\pi}=1-\frac{2i+3}{2}x+\frac{4i^2-3^2}{2^2}\frac{x^2}{2!}-\frac{4i^2-3^2}{2^2}\frac{2i+7}{2^2}\frac{x^3}{3!}+\dots(169).$$

This gives for $i=0$

$$\frac{2E}{\pi}=1+\tfrac{3}{2}x-\tfrac{9}{4}\frac{x^2}{2!}-\tfrac{9}{4}\tfrac{7}{2}\frac{x^3}{3!}+\ldots\ \ldots\ldots\ldots(170),$$

$$\frac{2I}{\pi}=1-\tfrac{3}{2}x-\tfrac{9}{4}\frac{x^2}{2!}+\tfrac{9}{4}\tfrac{7}{2}\frac{x^3}{3!}+\ldots\ \ldots\ldots\ldots(171).$$

A property of the four functions P_n, R_n, U_n, V_n; Jacobi's polynomials T, § 30.

§ 30. It is known that we have, for Legendre's function P_n, the following equation

$$\int_{-1}^{+1} P_m P_n dx=0\ \ldots\ldots\ldots\ldots\ldots\ldots(172),$$

for any m not equal to n.

The functions R_n, U_n, V_n possess similar properties, viz., we have

$$\left.\begin{aligned}\int_{-1}^{+1}\frac{U_m U_n}{1+x}\,dx&=0\\ \int_{-1}^{+1}\frac{V_n V_m}{1-x}\,dx&=0\\ \int_{-1}^{+1}\frac{R_n R_m}{(1+x)(1-x)}\,dx&=0\end{aligned}\right\}\begin{aligned}&\ldots\ldots\ldots\ldots\ldots\ldots\ldots(173),\\ &m \text{ not } =n\ \ldots\ldots\ (174),\\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots(175).\end{aligned}$$

Thus these four functions are particular cases of the functions T_m, T_n, so denoted by Jacobi, which possess the property

$$\int_{-1}^{+1}\frac{T_m T_n}{(1+x)^{\lambda}(1-x)^{\mu}}\,dx=0\ \ldots\ldots\ldots\ldots(176),$$

for m not equal to n. These functions were applied by Tchebicheff to the solution of the question of finding an integral algebraic function of given degree, deviating in certain given limits the least from zero. Our cases correspond exactly to those considered by Tchebicheff from a quite different point of view, viz. those corresponding to

$$\lambda=0,\quad \mu=0,$$

$$\lambda=1,\quad \mu=0,$$

$$\lambda=0,\quad \mu=1,$$

$$\lambda=1,\quad \mu=1,$$

which give respectively P_n, U_n, V_n and R_n.

Expressed in the variable h, we have generally

$$\int_0^1 \frac{T_m T_n}{h^\mu h'^\lambda}\, dh = 0 \text{ ....(177)},$$

and particularly

$$\int_0^1 P_m P_n dh = 0 \text{ ....(178)},$$

$$\int_0^1 \frac{U_m U_n}{h'}\, dh = 0 \text{ ....(179)},$$

$$\int_0^1 \frac{V_m V_n}{h}\, dh = 0 \text{ ....(180)},$$

$$\int_0^1 \frac{R_m R_n}{hh'}\, dh = 0 \text{ ....(181)},$$

and obviously also

$$\int_0^1 K_i K_j\, dh = 0 \text{ ....(182)},$$

$$\int_0^1 \frac{E_i E_j}{h'}\, dh = 0 \text{ ....(183)},$$

$$\int_0^1 \frac{I_i I_j}{h}\, dh = 0 \text{ ....(184)},$$

$$\int_0^1 \frac{G_i G_j}{hh'}\, dh = 0 \text{......................(185)};$$

we may write also

$$\int_0^1 p_m p_n dh = 0,\ \int_0^1 h' u_m u_n dh = 0,\ \int_0^1 h' v_m v_n dh = 0,\ \int_0^1 hh' v_m v_n dh = 0.$$

These equations may be proved in the following way. We have

$$\int_0^1 \frac{R_m R_n}{hh'}\, dh = \int_0^1 \frac{R_m}{hh'}\left(-\frac{hh'}{n}\frac{dP_n}{dh}\right) dh = -\frac{1}{n}\int_0^1 R_m dP_n$$

$$= -\frac{1}{n}\int_0^1 R_m P_n + \frac{1}{n}\int_0^1 P_n \frac{dR_m}{dh}\, dh.$$

But R_m vanishes at both limits; therefore

$$\int_0^1 \frac{R_m R_n}{hh'}\, dh = \frac{1}{n}\int_0^1 P_n \frac{dR_m}{dh}\, dh = \frac{m+1}{n}\int_0^1 P_n P_m dh = 0,$$

Q. E. D.

Having thus proved the formula for R we may find easily the corresponding expressions for U and V.

Thus the four functions appear once more to constitute one closed group of most intimately connected quantities completing each other. The generating function of any of them is included in the general expression

$$\frac{\{1+s+\sqrt{(1-2sx+s^2)}\}^{\lambda}\{1-s+\sqrt{(1-2sx+s^2)}\}^{\mu}}{\sqrt{(1-2sx+s^2)}} \quad \text{...(186)}.$$

Writing generally $G(F)$ for the generating function of F, we have

$$G(P_n)=\frac{1}{\sqrt{(1-2sx+s^2)}} \quad \text{............(187)},$$

$$G(R_n)=\frac{1-sx}{\sqrt{(1-2sx+s^2)}} \quad \text{............(188)},$$

$$G(U_n)=\frac{1+s}{\sqrt{(1-2sx+s^2)}} \quad \text{............(189)},$$

$$G(V_n)=\frac{1-s}{\sqrt{(1-2sx+s^2)}} \quad \text{............(190)}.$$

The numerators of the generating functions of U_n, V_n are respectively the maximum and minimum values of the denominator.

Another property of the same functions, § 31.

§ 31. It is also known from the theory of Legendre's functions that they may be represented as the n^{th} differential coefficient of x^2-1, viz.

$$P_n(x)=\frac{1}{2^n}\frac{1}{n!}\frac{d^n(x^2-1)^n}{dx^n} \quad \text{.........(191)},$$

or, putting $x=1-2h$, *i.e.* $dx=-2dh$,

$$P_n(h)=\frac{1}{n!}\frac{d^n[h(1-h)]^n}{dh^n} \quad \text{............(192)}.$$

Consider the analogous expressions

$$H_1=\frac{d^n}{dh^n}[h^n(1-h)^{n-1}],\quad H_2=\frac{d^n}{dh^n}[h^{n-1}(1-h)^n];$$

developing the expressions in brackets, we have

$$H_1 = \frac{d^n}{dh^n} \Sigma (-)^i \frac{(n-1)!}{(n-i-1)!\, i!} h^{n+1}$$

$$= \quad \Sigma (-)^i \frac{(n-1)!\,(n+i)!}{(n-i-1)!\, i!\, i!} h^i,$$

$$H_2 = \frac{d^n}{dh^n} \Sigma (-)^{i-1} \frac{n!}{(n-i-1)!\,(i+1)!} h^{n+i}$$

$$= \quad \Sigma (-)^{i-1} \frac{n!\,(n+i)!}{(n-i-1)!\,(i+1)!\, i!} h^i.$$

But we have seen that

$$c_n u_{n-1} = \Sigma (-)^i \frac{(n^2-1^2)\dots(n^2-i^2)}{(i!)^2} h^i$$

$$= \Sigma (-)^i \frac{(n+i)!}{(n-i-1)!\, i!\, i!} \frac{1}{n} h^i,$$

$$c_n v_{n-1} = \Sigma (-)^i \frac{(n^2-1^2)\dots(n^2-i^2)}{i!\,(i+1)!} h^i$$

$$= \Sigma (-)^i \frac{(n+i)!}{(n-i-1)!\, i!\,(i+1)!} \frac{1}{n} h^i,$$

thus
$$c_n u_{n-1} = \frac{d^n}{dh^n} [h^n (1-h)^{n-1}] \dots\dots\dots (193),$$

$$c_n v_{n-1} = (-)^n \frac{d^n}{dh^n} [h^{n-1} (1-h)^n] \dots\dots\dots (194),$$

where the constant factors are incorporated in the arbitrary constants c_n.

In the same way we shall find

$$c_n r_{n-1} = \frac{d^n}{dh^n} [h^{n-1} (1-h)^{n-1}] \quad \dots\dots\dots (195).$$

Thus the complete system involves the four products

$$h^n (1-h)^n, \quad h^n (1-h)^{n-1},$$
$$h^{n-1} (1-h)^n, \quad h^{n-1} (1-h)^{n-1},$$

and p_n, u_n, v_n, r_n are respectively proportional to the n^{th} differential coefficients of these quantities. We find for the functions P_n, U_n, V_n, R_n the elegant formulæ

$$c_n P_n = \frac{d^n}{dh^n}\{h^n(1-h)^n\} \quad \text{.........(196)},$$

$$c_n U_n = (1-h)\frac{d^n}{dh^n}\{h^n(1-h)^{n-1}\} \quad \text{.........(197)},$$

$$c_n V_n = h\,\frac{d^n}{dh^n}\{h^{n-1}(1-h)^n\} \quad \text{........(198)},$$

$$c_n R_n = h(1-h)\frac{d^n}{dh^n}\{h^{n-1}(1-h)^{n-1}\} \quad \text{......(199)},$$

and it is easy to transcribe these formulæ for x as the independent variable, writing $1+x=2h$, $1-x=2(1-h)$.

The notation of Weierstrass, § 32.

§ 32. Weierstrass uses, instead of K and E, the two functions ω, η, of which the expression shall be given hereafter, and instead of k^2, k'^2 the quantities e_1, e_2, e_3, or also g_2, g_3, λ, Δ, thus defined: e_1, e_2, e_3, are the roots of the cubic in the denominator of the integral

$$\int \frac{dy}{\{(y-e_1)(y-e_2)(y-e_3)\}} \quad \text{...........(200)},$$

g_2, g_3, are the invariants of this cubic

$$\tfrac{1}{4}g_2 = \tfrac{1}{2}(e_1^2+e_2^2+e_3^2) = -(e_1e_2+e_2e_3+e_3e_1) \quad \text{......(201)},$$

$$\tfrac{1}{4}g_3 = e_1e_2e_3 \quad \text{..................................(202)},$$

and

$$e_1+e_2+e_3=0 \quad \text{..................................(203)},$$

Δ its discriminant; thus

$$\Delta = g_2^3 - 27g_3^2 = 16(e_1-e_2)^2(e_2-e_3)^2(e_3-e_1)^2 \quad \text{...(204)}.$$

Lastly

$$\lambda = \frac{1}{e_1-e_3} \quad \text{.......................(205)},$$

Expressed in terms of the modulus

$$e_1 = \frac{2-k^2}{3\lambda},\ e_2 = \frac{2k^2-1}{3\lambda},\ e_3 = -\frac{1+k^2}{3\lambda} \quad \text{......(206)},$$

or $$e_1 = \frac{1+h'}{3\lambda},\ e_2 = \frac{h-h'}{3\lambda},\ e_3 = -\frac{1+h}{3\lambda} \text{......(207)},$$

whence also $$h = \frac{e_2-e_3}{e_1-e_3},\ h' = \frac{e_1-e_2}{e_1-e_3} \text{......................(208)}.$$

Then, with these definitions,

$$\omega = \sqrt{\lambda}\,.\,K \text{...(209)},$$

$$\eta = -e_1\omega + \sqrt{\lambda}\,.\,E = \sqrt{\lambda}\{E - \tfrac{1}{3}(1+h')K\} \text{....(210)},$$

or else $$K = \frac{\omega}{\sqrt{\lambda}} \text{..(211)},$$

$$E = \frac{\eta + e_1\omega}{\sqrt{\lambda}} \text{..(212)},$$

Whereas thus ω corresponds exactly to K, η is more similar to G of Dr. Glaisher.

Associative relation between $\omega\Delta^{\frac{1}{12}}$ *and* $\eta\Delta^{-\frac{1}{12}}$, § 33.

§ 33. Introduce a new quantity, the absolute invariant J,

$$J = \frac{g_2^{\ 3}}{\Delta} \text{........................(213)},$$

and write for shortness,

$$\omega\Delta^{\frac{1}{12}} = \omega_1,\ \eta\Delta^{\frac{1}{12}} = \eta_1 \text{...............(214)}.$$

Then $$\frac{d\omega_1}{dJ} = -\frac{1}{24\sqrt{3}}(J-1)^{-\frac{1}{2}}J^{-\frac{2}{3}}\eta_1 \text{.........(215)},$$

$$\frac{d\eta_1}{dJ} = \frac{1}{24\sqrt{3}}(J-1)^{-\frac{1}{2}}J^{-\frac{1}{3}}\omega_1 \text{.........(216)}.$$

Thus ω_1 and η_1 are associated functions, and the equations satisfied by them are

$$J(1-J)\frac{d^2\omega_1}{dJ^2} + \tfrac{1}{6}(4-7J)\frac{d\omega_1}{dJ} - \tfrac{1}{144}\omega_1 = 0 \text{...(217)},$$

$$J(1-J)\frac{d^2\eta_1}{dJ^2} + \tfrac{1}{6}(2-5J)\frac{d\eta_1}{dJ} - \tfrac{1}{144}\omega_1 = 0 \text{...(218)}.$$

The first of these equations is already known; it has been found by *Brims* (*Dorpater Festschrift* 1875: Ueber die Perioden der elliptischen Integrale. See also Halphen, *Traité des fonctions elliptiques* I. p. 314, whence I take the quotation).

Thus we may write

$$c_1\omega_1 = F(\tfrac{1}{12}, \tfrac{1}{12}, \tfrac{2}{3}, J) \quad \text{.........(219)},$$

$$c_1\eta_1 = F(-\tfrac{1}{12}, -\tfrac{1}{12}, \tfrac{1}{3}, J) \quad \text{.........(220)}.$$

Relation between ω and η, § 34.

§ 34. The quantities ω and η themselves are in a relation similar to the associative relation which exists between ω_1 and η_1. For if we denote by D the operator

$$D = 12g_3\frac{\partial}{\partial g_2} + \tfrac{2}{3}g_2^2\frac{\partial}{\partial g_3},$$

we shall have (Halphen, *loc cit*, pp. 307 and 308)

$$D\omega = -2\eta \quad \text{.....................(221)},$$

$$D\eta = \tfrac{1}{6}g_2\omega \quad \text{.....................(222)},$$

the operator D having the following property

$$D\Delta = 0 \quad \text{.........................(223)}.$$

Differential equation of the first order satisfied by the ratio $\dfrac{\omega_1}{\eta_1}$, § 35.

§ 35. I have shown in § 10 that the ratio of any two associated functions satisfies a differential equation of the first order and second degree. Since K and G, and also E and I, are associated functions, they satisfy a similar equation, as I have shown in my paper quoted above. Again as ω_1 and η_1 are associated functions they also must satisfy a differential equation of the said form. And in fact, if we put

$$u = \frac{\omega_1}{\eta_1} = \frac{\omega}{\eta}\Delta^{\frac{1}{6}} \quad \text{...............(224)},$$

$$\frac{1}{u} = v = \frac{\eta_1}{\omega_1} = \frac{\eta}{\omega}\Delta^{-\frac{1}{6}} \quad \text{...............(225)}.$$

we find the differential equations

$$24\sqrt{3}\,(J-1)^{\frac{1}{2}}J^{\frac{1}{3}}\frac{du}{dy} + u^2 + 12J^{-\frac{1}{3}} = 0 \quad \text{......(226)},$$

$$24\sqrt{3}\,(J-1)^{\frac{1}{2}}J^{\frac{1}{3}}\frac{dv}{dy} - 12J^{-\frac{1}{3}}v^2 - 1 = 0 \quad \text{...(227)}.^*$$

* The differential equation of the first sides satisfied by the ratios $\dfrac{K}{G}$ and $\dfrac{E}{I}$ has been given in my paper quoted above.

III.

Geometrical illustration of some of the formulæ contained in this paper.

Positive and negative association, § 36.

§ 36. The associative property defined in the first part of this paper admits of a very elegant geometrical illustration.

Suppose that we have transformed the equation satisfied by y to that independent variable which gives for this equation the form indicated in § 6, *i.e.*

$$L\Lambda = -c.$$

We see from the eq. (26) that the new variable will be real only if the sign of c be the same as that of $\frac{Q}{N}$. Thus we have two different cases to consider $c > 0$ or $c < 0$, and we may put

$$c = \pm 1.$$

The two fundamental forms thus are:

$$\frac{1}{\Lambda}\frac{d^2y}{dx^2} - \frac{1}{\Lambda^2}\frac{d\Lambda}{dx}\frac{dy}{dx} - Ly = 0,$$

giving
$$\frac{d^2y}{dx^2} + \frac{1}{L}\frac{dL}{dx}\frac{dy}{dx} - y = 0 \quad \text{..............(228)},$$

or
$$\frac{d^2y}{dx^2} + \frac{1}{L}\frac{dL}{dx}\frac{dy}{dx} + y = 0 \quad \text{..............(229)},$$

and for the associated function

$$\frac{d^2\eta}{dx^2} + \frac{1}{\Lambda}\frac{d\Lambda}{dx}\frac{d\eta}{dx} - \eta = 0 \quad \text{..............(230)},$$

or
$$\frac{d^2\eta}{dx^2} + \frac{1}{\Lambda}\frac{d\Lambda}{dx}\frac{d\eta}{dx} + \eta = 0 \quad \text{..............(231)},$$

where $L\Lambda = +1$ in the first case and -1 in the second.

We may write the last equations also in the following form:

$$\frac{d^2\eta}{dx^2} - \frac{1}{L}\frac{dL}{dx}\frac{d\eta}{dx} - \eta = 0,$$

$$\frac{d^2\eta}{dx^2} - \frac{1}{L}\frac{dL}{dx}\frac{d\eta}{dx} + \eta = 0,$$

in which they involve the same function L, which appears in the equations satisfied by y.

The associative relations between y and η are in this case

$$\frac{dy}{dx} = \frac{1}{L}\eta,$$

$$\frac{d\eta}{dx} = Ly,$$

or

$$\frac{dy}{dx} = -\frac{1}{L}\eta,$$

$$\frac{d\eta}{dx} = Ly.$$

Now we have in the first case

$$\frac{dy}{dx}\frac{d\eta}{dx} = y\eta.$$

and in the second

$$\frac{dy}{dx}\frac{d\eta}{dx} = -y\eta,$$

or

$$\frac{1}{y}\frac{dy}{dx}\cdot\frac{1}{\eta}\frac{d\eta}{dx} = \pm 1 \quad\text{..............(232).}$$

If we take x, y as the polar coordinates $(=\theta, r)$ of a curve in the plane, and x, η as the polar coordinates $(=\theta, \rho)$ of *the associated curve*, we shall have from this equation the following property of the two curves:

$$\frac{1}{r}\frac{dr}{d\theta}\cdot\frac{1}{\rho}\frac{d\rho}{d\theta} = \pm 1 \quad\text{..............(233).}$$

Now, if we denote by η the angle which the tangent at any point of the first curve makes with the radius vector, and by r the angle which the tangent of the corresponding point of the second curve makes with the same radius vector, we shall have

$$\tan m \tan\mu = \pm 1 \quad\text{.................(234),}$$

if we define the *corresponding* points to be those on the same *radius vector* (*i.e.* for which θ is the same).

This is the first case

$$\tan m = \cot\mu \quad\text{.................(235),}$$

and in the second

$$\tan m = -\cot\mu \quad\text{.................(236),}$$

i.e. in the latter case *the tangents at the corresponding points of two associated curves are at right angles*, and in the former case *the tangent of one of two associated curves and the normal at the corresponding point of the other make equal angles with the common radius vector.*

I shall call in the case $c = +1$ the two curves *positively* associated, in the second case $c = -1$ *negatively* associated.

Special case (i). *Two circles*, § 37.

§ 37. In the simplest case when $L = \Lambda = c$, we have seen in § 5 that the functions y, η are $\cos\theta$ and $\sin\theta$ respectively; thus, if we draw the curves

$$r = c\cos\theta,$$

$$\rho = \frac{1}{c}\sin\theta,$$

c being any constant, we shall have

$$\frac{1}{r}\frac{dr}{d\theta}\cdot\frac{1}{\rho}\frac{d\rho}{d\theta} = -1,$$

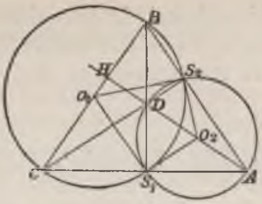
and the two curves are negatively associated.

These equations represent two circles, the centre of one of them being on the axis of x, that of the other on the axis of y. Thus the radii through one of the common points of the circles are at right angles. But then the radii through the second common point are also at right angles or the circles cut each other orthogonally. Thus we have the theorem:

If we draw a straight line through one of the points of intersection of two circles cutting each other orthogonally, this line shall meet the two circles in two other points such that the tangents at these points to the circles shall be at right angles to each other; or, in other words, *the radii through the points of intersection of this line with the two circles are at right angles.*

More generally, taking account also of the other points of intersection of these radii with the circles, and of the second point of intersection of the two circles, we may enunciate the following proposition:

If we draw through the centres of two circles cutting each other orthogonally any two straight lines at right angles, these lines shall intersect the circles in two pairs of points such that each pair of them shall be collinear with the corresponding point of intersection of the two circles.



Thus, in the figure the radii O_1H, O_2H drawn at right angles intersect the circles in the points B, C and A, D, and we have the collinear systems

$$AS_1C,\ CDS_2,\ BS_2A,\ BDS_1.$$

Special case (ii). *The hyperbolic sine and cosine*, § 38.

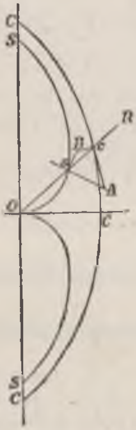
§ 38. In the same way we find the corresponding property for the case of the hyperbolic sine and cosine. The figure shows the curves

$$r = c\,(e^{\theta} + e^{-\theta}),$$

$$\rho = c\,(e^{\theta} - e^{-\theta}).$$

As in this case we have

$$\frac{1}{r}\frac{dr}{d\theta} \cdot \frac{1}{\rho}\frac{d\rho}{d\theta} = +1,$$



this equation shows that if we draw any radius vector such as OR, intersecting the hyperbolic sine curve in s and the hyperbolic cosine curve in c, then the tangent at c and the normal at s to the corresponding curves make equal angles with OR, or the triangle cAs is isosceles. The same is true for the triangle sBc, of which the sides sB, cB are respectively the tangent at s to the sinh-curve and the normal at c to the cosh-curve.

Special case (iii). *Legendre's Functions*, § 39.

§ 39. We have investigated in this paper some properties of the function R_n associated with Legendre's function P_n. We have seen that if θ be the independent variable the equations of associative relation are

$$\frac{dP_n}{d\theta} = -\frac{2nR_n}{\sin\theta\cos\theta},$$

$$\frac{dR_n}{d\theta} = 2(n+1)\sin\theta\cos\theta P_n,$$

whence
$$\frac{1}{P_n}\frac{dP_n}{d\theta}\cdot\frac{1}{R_n}\frac{dR_n}{d\theta} = -4n(n+1) \quad \text{......(237)}.$$

The right-hand side becomes equal to $+1$ for $n=-\frac{1}{2}$, *i.e.* when P_n, R_n become K and G respectively.

In the same way we have for our new functions U_n, V_n the relation

$$\frac{1}{U_n}\frac{dU_n}{d\theta}\cdot\frac{1}{V_n}\frac{dV_n}{d\theta} = -4n^2 \quad \text{............(238)},$$

which becomes equal to -1 for $n=\pm\frac{1}{2}$. But in this case the functions u_n, v_n themselves become E and I respectively. Thus we have to consider only these cases of our system of four functions P_n, R_n, U_n, V_n.

Special case (iv). *Elliptic Functions*, § 40.

§ 40. The considerations of the last section lead us to the following propositions.

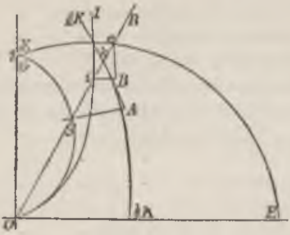
If we draw the curves represented by the equations

$$r = c_1K,\ \rho = c_2G,$$

$$r = c_1E,\ \rho = c_2I,$$

the modular angle θ being taken as the polar angle in polar coordinates, then (1) the tangent to one and the normal to the other of the curves (K) and (G) at points lying on the same radius vector shall make equal (or rather supplementary) angles with this radius vector, and (2) the tangents to the curves (E) and (I) shall be at right angles to each other. Or, in other words:

A radius vector through the origin constitutes (1) with the normal of one of the curves (K), (G) and the tangent to the other at the points of intersection an isosceles triangle, and (2) with the normals or tangents of the curves (E), (I) a right-angled triangle.



I have drawn on the figure the curves representing the four elliptic integrals K, G, E, I. The constant c has been taken equal to 1 for G, E, I, and to $\frac{1}{2}$ for K, in order to reduce the dimensions of the figure. With these constants the curves $\frac{1}{2}K$, E, I intersect in the same point.

The radius OR drawn through the origin O meets the four curves in the points g, i, k, e. The tangent kA at k to $\frac{1}{2}K$, and the normal gA at g to G meet in A, and gAk is an isosceles triangle. Again, the normals eB and iB at e and i to E and I meet in B and cut here at right angles.

CONTENTS.

I. THEORY OF ASSOCIATED FUNCTIONS..

II. APPLICATION OF THE THEORY OF ASSOCIATED FUNCTIONS TO A SPECIAL CASE.

III. GEOMETRICAL ILLUSTRATION OF SOME OF THE FORMULÆ CONTAINED IN THIS PAPER.

St. Petersburgh, *February*, 1890.