The product of the differences of these is

$$r-1!!\ n^{\frac{1}{2}r(r-1)}\phi(m_1)\,\phi(m_2)\,\ldots\,\phi(m_k)\,\zeta^{\frac{1}{2}}(m_1, m_2, \ldots, m_k),$$

where $\quad \phi(m)=m.m-n.m-2n \ldots m-(r-1)\,n.$

There being $r+k$ original quantities, this product is divisible by $r+k-1!!$, and hence

$$n^{\frac{1}{2}r(r-1)}\phi(m_1)\,\phi(m_2)\,\ldots\,\phi(m_k)\,\zeta^{\frac{1}{2}}(m_1, m_2, \ldots, m_k)$$

is divisible by $r!\ r+1!\ r+2!\ \ldots\ r+k-1!$.

This result reduces to that of § 1 when we make r zero; when we make r equal to unity, it gives that

$$m_1 m_2 \ldots m_k \zeta^{\frac{1}{2}}(m_1, m_2, \ldots, m_k)$$

is divisible by $k!!$.

Hence, *if k integers $m_1, m_2, \ldots, m_k$, be all prime to each of* $1, 2, 3, \ldots, k$, *or even such that an integer p can be found such that $m_1+p, m_2+p, \ldots, m_k+p$ are all prime to each of* $1, 2, 3, \ldots, k$, *then the product of their differences is divisible by* $k!!$.

THE NUMERICAL VALUE OF $\Pi i, =\Gamma(1+i)$.

By Prof. *Cayley*.

I do not know whether the numerical value of Πx for an imaginary value of x has ever been calculated; and I wish to calculate it for a simple case $x=i$.

We have

$$\frac{1}{\Pi z}=\left(1+\frac{z}{1}\right)$$

$$\left(1+\frac{z}{2}\right)e^{z\,\mathrm{hl}\frac{1}{2}}$$

$$\left(1+\frac{z}{3}\right)e^{z\,\mathrm{hl}\frac{2}{3}}$$

$$\vdots$$

$$\left(1+\frac{z}{s}\right)e^{z\,\mathrm{hl}\frac{s-1}{s}}$$

$$\vdots$$

where hl denotes the hyperbolic logarithm. Hence, in particular, $z=i$, we have

$$\frac{1}{\Pi i}=1+\frac{i}{1}$$
$$1+\frac{i}{2}.\cos \mathrm{hl}\tfrac{1}{2}+i\sin \mathrm{hl}\tfrac{1}{2}.$$
$$1+\frac{i}{3}.\cos \mathrm{hl}\tfrac{2}{3}+i\sin \mathrm{hl}\tfrac{2}{3}.$$
$$1+\frac{i}{4}.\cos \mathrm{hl}\tfrac{3}{4}+i\sin \mathrm{hl}\tfrac{3}{4}.$$
$$\vdots$$
$$=\sqrt{(1+1)}.\cos\theta_1+i\sin\theta_1.\cos\phi_1-i\sin\phi_1.$$
$$\sqrt{(1+\tfrac{1}{4})}.\cos\theta_2+i\sin\theta_2.\cos\phi_2-i\sin\phi_2.$$
$$\sqrt{(1+\tfrac{1}{9})}.\cos\theta_3+i\sin\theta_3.\cos\phi_2-i\sin\phi_3,$$
$$\vdots$$

($\phi_1=0$, and in the subsequent terms the imaginary part is taken with a negative sign in order to obtain positive values for ϕ_2, ϕ_3, &c.), $=\Omega(\cos\Theta+i\sin\Theta)$, if Ω be the modulus and Θ the sum $(\theta_1-\phi_1)+(\theta_2-\phi_2)+(\theta_3-\phi_3)+\dots$.

We have $\Omega_1=\sqrt{(1+1)}.\sqrt{(1+\tfrac{1}{4})}.\sqrt{(1+\tfrac{1}{9})}\dots$, which may be calculated directly: the value of Ω admits, however, of a finite expression, viz. we have

$$\Omega^2=\frac{1}{\Pi i\,\Pi(-i)}=\frac{\sin\pi i}{\pi i}=\frac{e^{\pi}-e^{-\pi}}{2\pi},$$

the approximate numerical value is $\Omega=1\cdot9173$, viz. we have

$$e^{\pi}-e^{-\pi}=23\cdot141-\cdot043=23\cdot098:\ \log=1\cdot3635744,$$
$$-\log 2\pi=\bar{1}\cdot201819,\ \text{whence}\ \log\Omega^2=\cdot5653935,$$
$$\log\Omega=2826967,\ \text{or}\ \Omega=1\cdot9173.$$

We have $\tan\theta_1=1$, $\tan\theta_2=\tfrac{1}{2}$, $\tan\theta_3=\tfrac{1}{3}$, &c.,

also $$\phi_1=0,\ \phi_2=\frac{180^\circ}{M\pi}\log\tfrac{1}{2},\ \phi_3=\frac{180^\circ}{M\pi}\log\tfrac{2}{3},\ \text{\&c.},$$

where M is the modulus for the Briggian logarithms,

$M=\cdot4342944$ log $=\bar{1}\cdot6377843$,
$\pi=3\cdot1415926$ „ $=\cdot4971499$,
180 „ $=2\cdot2552755$,

whence $$\log\frac{180}{M\pi}=2\cdot1203383,\ \frac{180^\circ}{M\pi}=131^\circ\cdot9284.$$

We hence calculate the succession of values of θ and ϕ as follows:

θ	tan	arc
1	1	45°
2	·5	26 34′
3	·3333333	18 26
4	·25	14 2
5	·2	11 19
6	·1666666	9 28
7	·1428571	8 8
8	·125	7 8
9	·1111111	6 20
10	·1	5 43

ϕ	131°.93 ×			$\theta - \phi =$	
1			= 0	1	45°
2	log 1/2 =	·3010300 =	39° 43	2	− 13° 9′
3	2/3	·1760913	23 14′	3	4 48
4	3/4	·1249387	16 29	4	2 27
5	4/5	·0969100	12 47	5	1 28
6	5/6	·0791813	10 26	6	0 58
7	6/7	·0669467	8 50	7	0 42
8	7/8	·0579920	7 39	8	0 31
9	8/9	·0511525	6 44	9	0 24
10	9/10	·0457575	6 2	10	0 19

The sum of all the negative arcs $\theta_2 - \phi_2$, $\theta_3 - \phi_3$, ... as far as calculated, that is up to $\theta_{10} - \phi_{10}$ is $= 24°46'$, or, writing x for the sum of the remaining arcs $\theta_{11} - \phi_{11}$ to infinity, we have

$$\frac{1}{\Pi i} = 1{\cdot}9173\,(\cos\Theta + i\sin\Theta),$$

where $\Theta = 45° - 24°.46' - x, = 20°14' - x.$

It would not be difficult to calculate a limit to the value of x.