NOTE ON FUNCTIONS OF A REAL VARIABLE.

By *W. Burnside.*

In illustration of the properties of functions of a single variable several examples are known, shewing that for real values of the variable the function may be finite and continuous and yet may not possess a differential coefficient. One of the best known is, perhaps the function

$$\sum_0^\infty b^n \cos(a^n\theta),$$

where a and b are real quantities satisfying certain inequalities. This example is due to Weierstrass, while in the second volume of his collected works Schwartz has given another of a totally different nature.

These functions, since they do not possess derivatives, cannot obviously be expanded in a series of positive powers of $x-x_0$ for any real value whatever of x_0. An example is here offered of a function of a real variable which, while finite, continuous, and possessing derivatives, yet is incapable of being expanded in a positive power-series.

Lemma. If ε is a given irrational number (that is, if no such equation as $\varepsilon = P/Q$ holds, where P and Q are integers) an infinite series of positive integers m, m', m'', ... can be found such that the fractional parts of $m\varepsilon$, $m'\varepsilon$, $m''\varepsilon$, ... shall differ by less than any assigned difference δ from a given proper fraction P/Q.

Let ε be converted into an infinite continued fraction by carrying out a process exactly similar to that by which a quadratic surd is converted into a periodic continued fraction, and let p/q be one of the convergents.

Consecutive integers ϖ and $\varpi+1$ can always be determined such that

$$\frac{\varpi}{q} < \frac{P}{Q} < \frac{\varpi+1}{q},$$

and since p, q are relatively prime, m can be chosen so that

$$mp \equiv \varpi \pmod{q}.$$

Now $$\varepsilon \sim \frac{p}{q} < \frac{1}{q^2},$$

therefore $$m\varepsilon \sim \frac{mp}{q} < \frac{m}{q^2} < \frac{1}{q},$$

and $$\text{frac.}\,(m\varepsilon) \sim \frac{\varpi}{q} < \frac{1}{q}.$$

But $$\frac{P}{Q} \sim \frac{\varpi}{q} < \frac{1}{q}.$$

Hence $$\text{frac.}\,(m\varepsilon) \sim \frac{P}{Q} < \frac{2}{q},$$

and if p/q has been chosen so that $q\delta > 2$, m will be one integer satisfying the required condition. That there is an infinite number may be shewn as follows. If p'/q' is any other convergent,

$$\text{frac.}\,(q'\varepsilon) \sim 0 < \frac{1}{q'}.$$

Now clearly q and q' may be so chosen that

$$\text{frac.}\,(m\varepsilon) > \frac{P}{Q} \text{ and frac.}\,(q'\varepsilon) > 0,$$

whence $$\text{frac.}\,(m\varepsilon) + \text{frac.}\,(q'\varepsilon) < \frac{P}{Q} + \frac{2}{q} + \frac{1}{q'}.$$

But $$\text{frac.}\,\{(m+q')\,\varepsilon\} = \text{frac.}\,(m\varepsilon) + \text{frac.}\,(q'\varepsilon),$$

or $$\text{frac.}\,(m\varepsilon) + \text{frac.}\,(q'\varepsilon) - 1,$$

and when q and q' are sufficiently great the latter alternative is impossible, and therefore

$$\text{frac.}\,\{(m+q')\,\varepsilon\} - \frac{P}{Q} < \frac{2}{q} + \frac{1}{q'}.$$

Hence, if q, q' have been chosen so that

$$\delta > \frac{2}{q} + \frac{1}{q'},$$

and if $\frac{p''}{q''}$, $\frac{p'''}{q'''}$, ... are successive convergents, m, $m+q'$, $m+q''$, ... all satisfy the required conditions.

If now α is such that α/π is not a rational fraction, it follows at once from the preceding Lemma that an infinite series of integers m, m', ... can be found, such that $\tan m\alpha$,

$\tan m'\alpha$, ... shall differ by less than any assigned difference from $\tan\frac{P\pi}{Q}$, and therefore also from any given real quantity.

Consider now the function of a real variable given by

$$f(x) = \sum_0^\infty \frac{1}{n!}\,\frac{1}{1+a^{2n}(x-\tan n\alpha)^2},$$

where a is real and greater than unity.

Whatever real value x has, $f(x)$ is always finite, for it is less than $\sum_0^\infty \frac{1}{n!}$. Moreover the series defining $f(x)$ is uniformly convergent for all real values of x, so that $f(x)$ is continuous.

Again, $f(x)-f(y)$

$$= \sum_0^\infty \frac{1}{n!}\,\frac{(y-x)\,a^{2n}\,(x+y-2\tan n\alpha)}{\{1+a^{2n}(x-\tan n\alpha)^2\}\{1+a^{2n}(y-\tan n\alpha)^2\}}$$

$$= -(x-y)\sum_0^\infty \frac{a^n}{n!}\,\frac{p_n+q_n}{(1+p_n^2)(1+q_n^2)},$$

where $\quad p_n = a^n(x-\tan n\alpha),\;\; q_n = a^n(y-\tan n\alpha).$

Now, whatever real quantities p, q may be,

$$\frac{p+q}{(1+p^2)(1+q^2)} \ngtr \frac{3\sqrt{3}}{8},$$

so that the series for $\frac{f(x)-f(y)}{x-y}$ is uniformly convergent whatever real values x and y may have, and therefore the fraction $f(x)$ has a derivative. In a similar manner it may be shewn that it has a second differential coefficient, and so on.

Finally $f(x)$ can only be expanded in a series of positive powers of $x-x_0$, if each term in the series representing it is capable of such expansion. Now it is easily shewn that

$$\frac{1}{1+a^{2n}(x-\tan n\alpha)^2}$$

is capable of expansion in positive powers of $x-x_0$, provided that

$$(x-x_0)^2 < (x_0-\tan n\alpha)^2 + a^{-2n}.$$

But by the Lemma, whatever x_0 may be, an increasing series of positive integers m, ... can be found such that $(x_0 - \tan m\alpha)^2$, and therefore also $(x_0 - \tan m\alpha)^2 + a^{-2m}$, is less than any assignable quantity. Hence, there must be terms in the series for $f(x)$ which can only be expanded in negative powers of $x - x_0$, whatever value x_0 may have; and therefore for no real value of x_0 can $f(x)$ be expanded in a series of positive powers.

ON RICHELOT'S INTEGRAL OF THE DIFFERENTIAL EQUATION $\frac{dx}{\sqrt{X}} + \frac{dy}{\sqrt{Y}} = 0$.

By Prof. *Cayley*.

In the Memoir "Einige Neue Integralgleichungen des Jacobi'schen Systems Differentialgleichungen" *Crelle* t. 25 (1843) pp. 97-118, Richelot, working with the more general problem of a system of $n-1$ differential equations between n variables, obtains a result which in the particular case $n = 2$ (that is for the differential equation

$$\frac{dx}{\sqrt{X}} + \frac{dy}{\sqrt{Y}} = 0, \; X = a + bx + cx^2 + dx^3 + ex^4,$$

and Y the same function of y), is in effect as follows: an integral is

$$\left\{\frac{\sqrt{X}(\theta - y) - \sqrt{Y}(\theta - x)}{x - y}\right\}^2$$
$$= \square\,(\theta - x)(\theta - y) + \Theta + e(\theta - x)^2(\theta - y)^2,$$

where $\square$, θ are arbitrary constants, and Θ denotes the quartic function $a + b\theta + c\theta^2 + d\theta^3 + e\theta^4$;
viz. this is theorem 3, p. 107, taking therein $n = 2$, and writing θ, $\square$ for Richelot's α and const.

The peculiarity is that the integral contains apparently *two* arbitrary constants, and it is very interesting to show how these really reduce themselves to a single arbitrary constant.

Observe that on the right-hand side there are terms in θ^4, θ^3 whereas no such terms present themselves on the left-hand side. But by changing the constant $\square$, we can get rid of these terms, and so bring each side to contain only terms