12.

EXAMPLES OF THE DIALYTIC METHOD OF ELIMINATION AS APPLIED TO TERNARY SYSTEMS OF EQUATIONS.

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THIS method is of universal application, and at once enables us to reduce any case of elimination to the form of a problem, where that operation is to be effected between quantities linearly involved in the equations which contain them.

As applied to a binary system, $fx = 0$, $\phi x = 0$, the method furnishes a rule by which we may unfailingly arrive at *the determinant*, free from every species of irrelevancy, whether of a linear, factorial, or numerical kind.

The rule itself is given in the *Philosophical Magazine* (London and Edinburgh, Dec. 1840). The principle of the rule will be found correctly stated by Professor Richelot, of Königsberg, in a late number of *Crelle's Journal*, at the commencement of a memoir in Latin bordering on the same subject ("Nota ad Eliminationem pertinens").

My object at present is to supply a few instances of its application to ternary systems of equations.

Ex. 1. To eliminate x, y, z, between the three homogeneous equations

$$Ay^2 - 2C'xy + Bx^2 = 0, \qquad (1)$$

$$Bz^2 - 2A'yz + Cy^2 = 0, \qquad (2)$$

$$Cx^2 - 2B'zx + Az^2 = 0. \qquad (3)$$

Multiply the equations in order by $-z^2$, x^2, y^2, add together, and divide out by $2xy$; we obtain

$$C'z^2 + Cxy - A'xz - B'yz = 0. \qquad (4)$$

By similar processes we obtain

$$A'x^2 + Ayz - B'yx - C'zx = 0, \qquad (5)$$

$$B'y^2 + Bzx - C'zy - A'xy = 0. \qquad (6)$$

Between these six, treated as simple equations, the six functions of x, y, z, namely, x^2, y^2, z^2, xy, xz, yz, treated as *independent* of each other, may be eliminated; the results may be seen, by mere inspection, to come out

$$ABC(ABC - AB'^2 - BC'^2 - CA'^2 + 2A'B'C') = 0,$$

or rejecting the special (N.B. not *irrelevant*) factor ABC, we obtain

$$ABC - AB'^2 - BC'^2 - CA'^2 + 2A'B'C' = 0.$$

I may remark, that the equations (1), (2), (3), or (4), (5), (6), express the condition of

$$Ax^2 + By^2 + Cz^2 + 2A'yz + 2B'zx + 2C'xy,$$

having a factor $\lambda x + \mu y + \nu z$; a general symbolical formula of which I am in possession for determining in general the condition of any polynomial of any degree having a factor, furnishes me at once with either of the two systems indifferently. The aversion I felt to reject *either*, led me to employ both, and thus was the occasion of the Dialytic Principle of Solution manifesting itself.

Ex. 2.
$$Ax^2 + ayz + bzx + cxy = 0, \qquad (1)$$
$$My^2 + lyz + mzx + nxy = 0, \qquad (2)$$
$$Rz^2 + pyz + qzx + rxy = 0. \qquad (3)$$

Multiply equation (1) by $\beta y + \gamma z$, equations (2) and (3) by νz and κy respectively, and add the products together, we obtain terms of which y^2z and yz^2 are the only two into which x does not enter.

Make now the coefficients of each of these zero, and we have

$$a\gamma + l\nu + R\kappa = 0,$$
$$a\beta + M\nu + p\kappa = 0.$$

Let $\nu = a$, $\kappa = a$, then $\gamma = -(l + R)$, $\beta = -(M + p)$.

Hence, multiplying as directed, and then dividing out by x, we obtain

$$(m\nu + b\gamma) z^2 + (r\kappa + c\beta) y^2 + (b\beta + c\gamma + n\nu + q\kappa) yz + A\beta xy + A\gamma xz = 0,$$

or by substitution,

$$\{ra - c(M + p)\} y^2 + \{ma - b(l + R)\} z^2 + \{an + aq - b(M + p) - c(l + R)\} yz$$
$$- A(M + p) xy - A(M + p) xz = 0. \qquad (4)$$

Similarly, by preparing the equations so as to admit in turn of y and z as a divisor, we obtain

$$\{ma - l(R + b)\} z^2 + \{mr - n(A + q)\} x^2 + \{mc + mp - n(R + b) - l(A + y)\} xz$$
$$- M(R + b) yz - A(A + q) xy = 0, \qquad (5)$$

$$\{rm - q(A + n)\} x^2 + \{ra - p(M + c)\} y^2 + \{rl + rb - p(A + n) - q(M + c)\} xy$$
$$- R(A + n) xz - R(M + c) yz = 0. \qquad (6)$$

Between the six equations (1), (2), (3), (4), (5), (6), x^2, y^2, z^2, xy, xz, yz, may be eliminated; the result will be a function of nine letters {three out of each equation (1), (2), (3)} equated to zero. *Perhaps* the determinant may be found to contain a special factor of three letters; and if so, may be replaced by a simpler function of six letters only.

Ex. 3. To eliminate between the three general equations

$$Ax^2 + By^2 + Cz^2 + 2Dyz + 2Ezx + 2Fxy = 0,$$

$$Lx^2 + My^2 + Nz^2 + 2Pyz + 2Qzx + 2Rxy = 0,$$

$$fx + gy + hz = 0.$$

By virtue of *one* of the two canons which limit the forms in which the letters can appear combined in the determinant of a general system of equations, we know that the determinant in this case (freed of irrelevant factors) ought to be made up in every term of eight letters (powers being counted as repetitions), namely, (A, B, C, D, E, F) must enter in binary combinations, (L, M, N, P, Q, R) the same, whereas f, g, h must enter in *quaternary* combinations.

To obtain the determinant, write

$$Ax^2 + By^2 + Cz^2 + Dyz + Ezx + Fxy = 0, \qquad (1)$$

$$Lx^2 + My^2 + Nz^2 + Pyz + Qzx + Rxy = 0, \qquad (2)$$

$$fx^2 + gyx + hzx = 0, \qquad (3)$$

$$fxy + gy^2 + hzy = 0, \qquad (4)$$

$$fxz + gyz + hz^2 = 0. \qquad (5)$$

We want one equation more of *three* letters between x^2, y^2, z^2, xy, xz, yz. To obtain this, write

$$(Ax + Ez + Fy)\, x_1 + (By + Fx + Dz)\, y_1 + (Cz + Dy + Ex)\, z_1 = 0,$$

$$(Lx + Qz + Ry)\, x_1 + (My + Rx + Pz)\, y_1 + (Nz + Py + Qx)\, z_1 = 0,$$

$$fx_1 + \qquad gy_1 + \qquad hz_1 = 0.$$

Forget that $x_1 = x$, $y_1 = y$, $z_1 = z$, and eliminate x_1, y_1, z_1, we obtain

$$h\begin{Bmatrix}(Ax + Ez + Fy)(My + Rx + Pz)\\ -(By + Fx + Dz)(Lx + Qz + Ry)\end{Bmatrix}$$

$$+g\begin{Bmatrix}(Cz + Dy + Ex)(Lx + Qz + Ry)\\ -(Nz + Py + Qx)(Ax + Ez + Fy)\end{Bmatrix}$$

$$+f\begin{Bmatrix}(Nz + Py + Qx)(By + Fx + Dz)\\ -(Cz + Dy + Ex)(My + Rx + Pz)\end{Bmatrix} = 0.$$

This may be put under the form

$$\alpha x^2 + \beta y^2 + \gamma z^2 + \alpha' yz + \beta' zx + \gamma' xy = 0, \qquad (6)$$

where the coefficients are of the first order in respect to $f, g, h, L, M, N, P, Q, R, A, B, C, D, E, F$; in all of the third order.

Between the equations marked from (1) to (6), the process of linear elimination being gone through, we obtain as equated to zero a function of $5+3$, or of eight letters, two belonging to the first equation, two to the second, and four to the third; so that the determinant is clear of all factorial irrelevancy.

Ex. 4. To eliminate x, y, z between the three equations

$$Ax^2 + By^2 + Cz^2 + 2A'yz + 2B'zx + 2C'xy = 0,$$
$$Lx^2 + My^2 + Nz^2 + 2L'yz + 2M'zx + 2N'xy = 0,$$
$$Px^2 + Qy^2 + Rz^2 + 2P'yz + 2Q'zx + 2R'xy = 0.$$

Call these three equations $U=0$, $V=0$, $W=0$, respectively. Write

$xU=0,$	(1)	$yU=0,$	(2)	$zU=0,$	(3)
$xV=0,$	(4)	$yV=0,$	(5)	$zV=0,$	(6)
$xW=0,$	(7)	$yW=0,$	(8)	$zW=0.$	(9)

We have here nine unilateral equations: one more is wanted to enable us to eliminate *linearly* the ten quantities

$$x^3,\ y^3,\ z^3,\ x^2y,\ x^2z,\ xy^2,\ xz^2,\ xyz,\ y^2z,\ yz^2.$$

This tenth may be found by eliminating x, y, z between the three equations

$$x(Ax + B'z + C'y) + y(By + C'x + A'z) + z(Cz + A'y + B'x) = 0,$$
$$x(Lx + M'z + N'y) + y(My + N'x + L'z) + z(Nz + L'y + M'x) = 0,$$
$$x(Px + Q'z + R'y) + y(Qy + R'x + P'z) + z(Rz + P'y + Q'x) = 0;$$

for, by forgetting the relations between the bracketed and unbracketed letters, we obtain

$$(Ax + B'z + C'y)\begin{Bmatrix} (My + N'x + L'z)(Rz + P'y + Q'x) \\ -(Qy + R'x + P'z)(Nz + L'y + M'x) \end{Bmatrix}$$
$$+ \&c. + \&c. = 0,$$

which may be put under the form

$$\alpha x^3 + \beta y^3 + \gamma z^3 + \delta x^2 y + \ldots\ldots = 0^*. \qquad (10)$$

* We might dispense with a 10th equation, using the nine above given, to determine the ratios of the ten quantities involved to one another; and then by means of any such relations as

$$x^3y \times xy^3 = x^2y^2 \times x^2y^2, \text{ or } x^3 \times y^3 = x^2y \times xy^2, \ \&c.$$

obtain a determinant. But it is easy to see that this would be made up of terms, each containing literal combinations of the 18th order.

Again, we might use five out of the nine equations to obtain a new equation free from y^3, y^2z, yz^2, z^3; that is, containing x in every term: which being divided by x, and multiplied

By eliminating linearly between the equations marked from (1) to (10), we obtain as zero a quantity of the twelfth order in all, being of the fourth order in respect to the coefficients of each of the three equations, which is therefore the determinant in its simplest form.

I have purposely, in this brief paper, avoided discussing any theoretical question. I may take some other opportunity of enlarging upon several points which have hitherto been little considered in the theory of elimination, such as the Canons of Form,—the Doctrine of Special Factors,—the Method of Multipliers as extended to a system of any order,—the Connexion between the method of Multipliers and the Dialytic Process,—the Idea of Derivations and of Prime Derivatives extended to ultra-binary Systems. For the present I conclude with the expression of my best wishes for the continued success of this valuable Journal.

by y, or by z, would furnish a 10th equation no longer linearly involved in the 9 already found. The determinant, however, found in this way, would consist of 14-ary combinations of letters.

Finally, we might, instead of a system of ten equations, employ a system of 15, obtained by multiplying each of the given three by any 5 out of the 6 quantities x^2, y^2, z^2, xy, xz, yz; but the determinant, besides being not *totally* symmetrical, would contain combinations of the 15th order.

I may take this opportunity of just adverting to the fact, that the method in the text does in fact contain a solution of the equation

$$\lambda U + \mu V + \nu W = x^r y^s z^t,$$

where $r + s + t = 4$, and λ, μ, ν are functions of the second degree in regard to x, y, z to be determined.