20.

ON THE EQUATION IN NUMBERS $Ax^3 + By^3 + Cz^3 = Dxyz$, AND ITS ASSOCIATE SYSTEM OF EQUATIONS.

[*Philosophical Magazine*, XXXI. (1847), pp. 293—296.]

IN the last Number of this *Magazine* I gave an account of a remarkable transformation to which the equation

$$Ax^3 + By^3 + Cz^3 = Dxyz$$

is subject when certain conditions between the coefficients A, B, C, D are satisfied; which conditions I shall begin by expressing with more generality and precision than I was enabled to do in my former communication.

1. Two of the quantities A, B, C are to be to one another in the ratio of two cubes.

2. $27ABC - D^3$ must contain no positive prime factor whatever of the form $6n+1$. I erred in my former communication in not excluding cubic factors of this form.

3. If 2^m is the highest power of 2 which enters into ABC, and 2^n the highest power of 2 which enters into D, then either m must be of the form $3n \pm 1$, or if not, then m must be greater than $3n$.

These three conditions being satisfied, the given equation can always be transformed into another,

$$A'u^3 + B'v^3 + C'w^3 = D'uvw,$$

where

$$A'B'C' = ABC, \quad D' = D, \quad uvw = \text{a factor of } z.$$

The consequence of this is, as stated in my former paper, that wherever A, B, C, D, besides satisfying the conditions above stated, are taken so as likewise to satisfy the condition,—firstly, of ABC being equal to $2^{3m\pm1}$, or secondly, of ABC being equal to $2^{3m\pm1} . p^{3n\pm1}$, provided in the second case that ABC is of the form $9m \pm 1$, and that D is divisible by 9, p being in both cases a prime, then the given equation will be *generally* insoluble. And I am now enabled to add that the *only* solution of which it will in any case admit, is the solitary one found by making two of the terms Ax^3, By^3, Cz^3 equal to one another; so that, for instance, if the given equation should be of the form

$$x^3 + y^3 + ABCz^3 = Dxyz,$$

then the above conditions being satisfied, the one solitary solution of which the equation can possibly admit, is $x = 1$, $y = 1$,

$$Az^3 - Dz + 2 = 0,$$

which may or may not have possible roots. I call this a *solitary* or singular solution, because it exists alone and no other solution can be deduced from it; whereas in general I shall show that any one solution of the equation

$$Ax^3 + By^3 + Cz^3 = Dxyz$$

can be made to furnish an infinity of other solutions independent of the one supposed given, that is, not reducible thereto by expelling a common factor from the new system of values of x, y, z deduced from the given system.

The following is the Theorem of Derivation in question:

Let

$$A\alpha^3 + B\beta^3 + C\gamma^3 = D\alpha\beta\gamma.$$

Then if we write

$$F = A\alpha^3, \quad G = B\beta^3, \quad H = C\gamma^3,$$

and make

$$x = F^2G + G^2H + H^2F - 3FGH,$$

$$y = FG^2 + GH^2 + HF^2 - 3FGH,$$

$$z = \frac{1}{D}\{F^3 + G^3 + H^3 - 3FGH\},$$

or

$$= \alpha\beta\gamma\{F^2 + G^2 + H^2 - FG - FH - GH\},$$

we shall have

$$x^3 + y^3 + ABCz^3 = Dxyz.$$

I am hence enabled to show that whenever $x^3 + y^3 + Az^3 = Dxyz$ is insoluble, there will be a whole family of allied equations equally insoluble. For instance, because $x^3 + y^3 + z^3 = 0$ is insoluble in integer numbers, I know likewise that

$$x^6 + y^6 + z^6 = x^3y^3 + x^3z^3 + y^3z^3$$

$$x^6 + y^6 + z^6 = x^3y^3 + x^3z^3 - 2y^3z^3$$

are each equally insoluble.

In fact

$$\begin{aligned}(x^3+y^3+z^3) &\times (x^6+y^6+z^6-x^3y^3-x^3z^3-y^3z^3)\\ &\times (x^6+y^6+z^6-x^3y^3-x^3z^3+2y^3z^3)\\ &\times (x^6+y^6+z^6-y^3z^3-y^3x^3+2x^3z^3)\\ &\times (x^6+y^6+z^6-x^3z^3-z^3y^3+2y^3x^3)\\ &= u^3+v^3+w^3,\end{aligned}$$

where u, v, w are rational integral functions of x, y, z.

Hence each of the factors must be incapable of becoming zero*.

As a particular instance of my general theory of transformation and elevation, take the equation

$$x^3+y^3+2z^3 = Mxyz.$$

Then, with the exception of the singular or solitary solution $x=1$, $y=1$, of which I take no account, I am able to affirm that for all values of M between 7 and -6, both inclusive, with the exception of $M=-2$, the equation is insoluble in integer numbers.

Take now the equation where $M=-2$, namely

$$x^3+y^3+2z^3+2xyz=0.$$

One particular solution of this is

$$x=1, \quad y=-1, \quad z=1.$$

Another, which I shall call the second†, is

$$x=1, \quad y=3, \quad z=-2.$$

From the first solution I can deduce in succession the following:

$$\begin{array}{lll} x=11, & y=5, & z=-7, \\ x=-793269121, & y=1179490001, & z=-1189735855, \\ \quad\&c. & \quad\&c. & \quad\&c. \end{array}$$

From the second,

$$\begin{array}{lll} x=-10085, & y=8921, & z=-8442, \\ x=\&c. & y=\&c. & z=\&c. \end{array}$$

As another example, take the equation

$$x^3+y^3+6z^3=6xyz.$$

* It is however sufficiently evident from their intrinsic form, which may be reduced to $\frac{1}{4}(M^2+3N^2)$, that this impossibility exists for all the factors except the first.

† See Postscript.

One solution of the transformed equation

$$u^3 + 2v^3 + 3w^3 = 6uvw$$

is evidently

$$u = 1, \quad v = 1, \quad w = 1.$$

Hence I can deduce an infinite series of solutions of the given equation, of which the first in order of ascent will be

$$x = 5, \quad y = 7, \quad z = 3.$$

Again, the lowest possible solution in integers of the equation

$$x^3 + y^3 + 6z^3 = 0$$

will be

$$x = 17, \quad y = 37, \quad z = -21.$$

The equation

$$x^3 + y^3 + 9z^3 = 0$$

admits of the solutions

$$x = 1, \quad y = 2, \quad z = -1,$$
$$x = -271, \quad y = 919, \quad z = -438.$$

I trust that my readers will do me the justice to believe that I am in possession of a strict demonstration of all that has been here advanced without proof. Certain of the writer's friends on the continent have, in their comments upon one of his former papers which appeared in this *Magazine*, complimented his powers of divination at the expense of his judgment, in rather gratuitously assuming that the author of the Theory of Elimination was unprovided with the demonstrations, which he was too inert or too beset with worldly cares and distractions to present to the public in a sufficiently digested form. The proof of whatever has been here advanced exists not merely as a conception of the author's mind, but fairly drawn out in writing, and in a form fit for publication.

P.S. It must not be supposed that the two primary or basic solutions above given of the equation

$$x^3 + y^3 + 2z^3 + 2xyz = 0,$$

namely,

$$x = 1, \quad y = -1, \quad z = 1,$$
$$x = 1, \quad y = 3, \quad z = -2,$$

are independent of one another. The second may be derived from the first, as I shall show in a future communication. In fact there exist *three* independent processes, by combining which together, one particular solution may be made to give rise to an infinite series of infinite series of infinite series of correlated solutions, which it may possibly be discovered contain between them the *general* complete solution of the equation

$$x^3 + y^3 + Az^3 = Dxyz.$$