Arch. Mech., 38, 4, pp. 369–382, Warszawa 1986

Microstresses in homogenization

W. S. BARAŃSKI (ŁÓDŹ)

THE PAPER concerns the problem of homogenization of nonlinearly elastic, periodic composites within the framework of the geometrically linear theory. It is assumed that the elasticity moduli of the considered composite are upper and lower bounded. A new proof of the corresponding homogenization theorem is given. Strongly convergent approximations for deformations and stresses are found. The proposed method of homogenization analysis aims also at making clear the mechanical principles of the homogenization theory. This is achieved by adapting the old concepts of macroscopic and microscopic mechanical fields.

Praca dotyczy zagadnienia homogenizacji nieliniowo sprężystego, periodycznego kompozytu, rozpatrywanego w ramach teorii geometrycznie liniowej. Założono, że moduły sprężystości rozpatrywanego kompozytu są ograniczone z góry i dołu. Podano nowy dowód odpowiedniego twierdzenia homogenizacyjnego. Znaleziono silnie zbieżne przybliżenia odkształceń i naprężeń. Zaproponowana metoda analizy homogenizacyjnej ma również na celu wyjaśnienie mechanicznych podstaw teorii homogenizacji. Jest to osiągnięte poprzez zaadaptowanie starych koncepcji wielkości makro- i mikroskopowych.

Работа касается задачи гомогенизации нелинейно упругого, периодического композита рассматриваемого в рамках геометрически линейной теории. Предположено, что модули упругости рассматриваемого композита ограничены сверху и снизу. Приведено новое доказательство соответствующей гомогенизационной теоремы. Найдены сильно сходящиеся приближения деформаций и напряжений. Предложенный метод гомогенизационного анализа имеет тоже целью выяснение механических основ теории гомогенизации. Это достигнуто путем приспособления старых концепций макро- и микроскопических величин.

1. Introduction

RECENT developments in the field of the homogenization theory have made clear the mathematical foundations for a deterministic approach to the mechanics of composite materials. The central problem in this theory is to find an accurate description of the macroscopic behaviour of nonhomogeneous materials. For its basic concept we refer to [5, 13, 25, 26]. Nowadays the theory develops in many different directions and successfully violates the range of linear problems [2, 3, 6, 7, 9, 10, 12, 17, 20, 22–24, 26, 27]. Despite of this, there is still a lot of unsolved questions concerning, first of all, the effects of degradation of the material. Moreover, there has not been a satisfactory answer as yet to the question of finding strongly convergent approximations for nonlinear problems.

The present paper aims at establishing a new method of homogenization analysis, which is as general as possible, and at examining the method in the case of physically nonlinear elasticity. The considered particular case, analysed previously by Suquet under somehow stronger assumptions, is relatively simple as compared with the generality of the method which seems to be applicable to many other problems including quasi-static and dynamic problems for viscous, plastic, cracked and degradating composites. The case of physically nonlinear elasticity has been chosen for simplicity of the presentation.

The proposed method of homogenization analysis rests on the concept of microfields. To present it, let us consider a composite occupying a domain of size 1, and having periodic structure of dimension ε. Obviously, the solution of any mechanical problem corresponding to this composite depends on ε and on l, and can be denoted by $s_{\varepsilon l}$. From the practical point of view we are interested in the analysis of $s_{\varepsilon l}$ when ε is very small with respect to l. It is realized in the homogenization theory by introducing the concept of macroscopic fields defined as the limit of $s_{\varepsilon l}$ as ε tends to zero with a fixed l. It has occurred that such defined macroscopic fields can be successfully analysed for many mechanical and physical problems, providing us with a mathematically clear concept of macroscopic properties of composite materials. However, the analysis is somehow artificial from the point of view of mechanics. To make it more natural, we shall define microscopic fields as the limit of $s_{\varepsilon l}$ as 1 converges to infinity with a fixed ε; this hence also realizes the postulate that the structure of nonhomogenity is fine. However, the picture obtained in such a realization is quite different. The material remains nonhomogeneous, the body becomes infinite, and the most important fact — the limit depends essentially on the choice of the fixed point of a collection of mappings extending the domain occupied by the composite to the whole Euclidean space. Hence we should define the microscopic fields as functions of two variables.

For certain technical resons we shall not use the large parameter l. Instead of this, we shall introduce local coordinate systems $\{y\}$ by

$$x = r_\varepsilon + \varepsilon y$$

with r_ε pointing at centres of periodicity cells. We define the microscopic fields as the limit, as ε converges to zero, of the solution $s_{\varepsilon l}$ considered as functions of local coordinates.

The paper concentrates on the mechanical aspects of the proposed method of homogenization analysis restricting the mathematical details to very essential ones. According to this concept, we shall not present detailed proofs of the obtained results indicating only the general line of the proceeding.

2. Notation

Let R^n denote the n-dimensional Euclidean space with the inner product denoted by the dot $\cdot$, and the norm denoted by $|\ |$. We shall denote the vectors in R^n by x, y, z. Let S^n denote the space of symmetric second order tensors on R^n with the same notation for the inner product and for the norm.

Let $Y = (y_1, \ldots, y_n)$ be a given set of n linearly independent vectors in R^n. We assume the set Y to be uniquely determined by the structure of the considered composite. We define the basic cell

$$C = C_Y = \left\{ \sum_{i=1}^{n} \lambda_i y_i : \lambda_i \in]-1/2,\ 1/2[\quad \text{for} \quad i = 1, \ldots, n \right\},$$

and the basic net

$$N = N_Y = \left\{ \sum_{i=1}^{n} \lambda_i y_i : \lambda_i = 0, \pm 1, \pm 2, \ldots \quad \text{for} \quad i = 1, \ldots n \right\}.$$

A function f defined on R^n will be called Y-periodic if

$$f(x+y) = f(x), \quad \text{for all} \quad y \in N_Y \quad \text{a.e.} \quad x \in R^n.$$

For a scalar $\alpha > 0$, a vector $x \in R^n$ and subsets A, B of R^n we define

$$\alpha A = \{\alpha y : y \in A\}, \quad x + A = \{x + y : y \in A\},$$
$$A + B = \{y + z : y \in A, z \in B\}.$$

For each subset A of R^n we denote its Lebesgue measure by means A.

In the paper we shall deal with several Banach spaces. In general, we shall denote the norm on a Banach space X by $\| \; ; X\|$. However, in the case of the space $L^2(\Omega; W)$ of square integrable mappings of a domain Ω in R^n into a finite dimensional Hilbert space W, we shall use simply $\| \; \|_\Omega$ for the corresponding norm. We shall simplify also the notation for the norm on the space $H^1(\Omega; W)$ of square integrable mappings of $\Omega \subset R^n$ into a finite dimensional Hilbert space W with square integrable first order distributional derivatives denoting the norm by $\| \; \|_{\Omega; 1}$.

Finally, we shall present the notation used for locally convex function spaces. Let us assume that $X(\Omega)$ is $L^2(\Omega; W)$ or $H^1(\Omega; W)$ with a finite dimensional Hilbert space W and some $\Omega \subset R^n$. Then

$$X_{\text{loc}}(R^n) = \{f; R^n \to W : f|_\omega \in X(\omega) \text{ for all open bounded } \omega \subset R^n\},$$
$$L^2(\Omega; X_{\text{loc}}(R^n)) = \{f; (\Omega \times R^n) \to W : f|_{\Omega \times \omega} \in L^2(\Omega; X(\omega))$$
$$\text{for all open bounded } \omega \text{ in } R^n\}.$$

3. Variational formulation of the equilibrium problem

Let U^{ad}, called here a manifold of admissible displacement fields, denote the set of displacement fields from $H^1(\Omega; R^n)$ satisfying a given system of kinematical boundary conditions imposed on the considered family of composites with $\Omega \subset R^n$ being a fixed domain.

We define the space of admissible displacement variations $V^{\text{ad}} = \{u - v : u, v \in U^{\text{ad}}\}$.

Let us consider the following set of equilibrium problems:

$P1_\varepsilon$. For give $\varepsilon \in]0, 1]$, $b_\varepsilon \in L^2(\Omega; R^n)$ and $t \in L^2(\partial\Omega; R^n)$ find:

(i) displacement vector field $u_\varepsilon \in U^{\text{ad}}$,

(ii) deformation tensor field $E_\varepsilon \in L^2(\Omega; S^n)$,

(iii) stress tensor field $T_\varepsilon \in L^2(\Omega; S^n)$ such that

$$E_\varepsilon(x) = Du_\varepsilon(x) = \text{sym}(\nabla u_\varepsilon(x)), \quad \text{s.e.} \quad x \in \Omega, \tag{3.1}$$

$$T_\varepsilon(x) = \mathcal{T}^\varepsilon(x, E_\varepsilon(x)) = \mathcal{T}(x/\varepsilon, E_\varepsilon(x)), \quad \text{a.e.} \quad x \in \Omega, \tag{3.2}$$

$$\int_\Omega Dv \cdot T_\varepsilon dx = \int_{\partial\Omega} v \cdot t\, ds + \int_\Omega v \cdot b_\varepsilon dx, \quad \forall v \in V^{ad}, \tag{3.3}$$

where ∇ denotes the distributional gradient on Ω.

We assume that the domain Ω, the manifold of admissible displacement fields U^{ad}, the constitutive function $\mathcal{T}$, the boundary tractions t and the volume forces b_ε fulfill the following assumptions:

A1. The domain Ω is open and bounded in R^n, has modified cone property ([1] p. 91) and has finite perimeter ([15] p. 474).

A2. The manifold U^{ad} of admissible displacement fields is a closed linear manifold in $H^1(\Omega; R^n)$ such that the induced space of admissible displacement variations V^{ad} includes $H^1_0(\Omega; R^n)$.

A3. The constitutive function $\mathcal{T}(y, E)$ is Y-periodic and measurable in y for all fixed $E \in S^n$.

A4. There exist positive constants c_1 and c_2 such that

$$(E_2 - E_1) \cdot [\mathcal{T}(y, E_2) - \mathcal{T}(y, E_1)] \geqslant c_1 |E_2 - E_1|^2$$

and

$$|\mathcal{T}(y, E_2) - \mathcal{T}(y, E_1)| \leqslant c_2 |E_2 - E_1|, \quad \forall E_1, E_2 \in S^n, \quad \text{a.e.} \quad y \in R^n.$$

A5. The field $\mathcal{T}(\cdot, 0)$ of initial stresses belongs to $L^2_{loc}(R^n; S^n)$.

A6. For each $\varepsilon \in]0, 1]$ the work done by external forces vanishes on the space of admissible rigid body displacement variations, i.e.

$$\int_\Omega v \cdot b_\varepsilon dx + \int_{\partial\Omega} v \cdot t\, ds = 0$$

for all v from

$$N^{ad} = \{v \in V^{ad} : Dv = 0\}.$$

A7. There exists $p > n$, $p \geqslant 2$ such that

$$b_\varepsilon \to b_0 \quad \text{in} \quad L^p(\Omega; R^n) \quad \text{weakly as} \quad \varepsilon \to 0.$$

Using the standard methods of analysis of quasi-linear elliptic equations [8, 19, 21], one can prove that the assumptions A1–A6 are sufficient for the existence of solutions for P^1_ε. Obviously, the solutions in deformations and stresses are unique, and the displacement solutions are the unique modulo N^{ad}. To avoid this drawback, we introduce the factor space ([28] p. 59)

$$H^1(\Omega; R^n)/N^{ad} = \{\tilde{v} = v + N^{ad} : v \in H^1(\Omega; R^n)\}$$

with the factor norm

$$\|\tilde{v}\|^{\sim}_{\Omega;1} = \inf_{v \in \tilde{v}} \|v\|_{\Omega;1}.$$

Consequently, the displacement solution $\tilde{u}_\varepsilon$ can be considered as defined uniquely. Moreover, the solutions can be proved to be bounded, i.e.

$$\max(\|\tilde{u}_\varepsilon\|^{\sim}_{\Omega;1}, \|E_\varepsilon\|_\Omega, \|T_\varepsilon\|_\Omega) \leqslant c \tag{3.4}$$

with c independent of ε.

In the further analysis we shall consider the deformation and the stress fields as defined on the whole R^n with zero value outside Ω. The displacement fields $\tilde{u}_\varepsilon$ will also be considered as defined on R^n but with values outside Ω obtained by applying an appropriate continuous extension operator. The existence of such an operation is stated in the Calderon extension theorem ([1] p. 91).

4. Macroscopic fields

From the practical point of view we are interested in the analysis of solutions of P_ε^1 in the case of small ε. Hence, it is interesting to analyse the limit of solutions of P_ε^1 as ε tends to zero. Therefore it is rational to define the macroscopic fields as the appropriate limits of $\tilde{u}_\varepsilon$, E_ε, T_ε. We shall prove in Sect. 11 that such limits exist in certain weak topologies. Now we can write only formally

$$\tilde{u}_m = \lim_{\varepsilon\to 0}\tilde{u}_\varepsilon, \qquad E_m = \lim_{\varepsilon\to 0} E_\varepsilon, \qquad T_m = \lim_{\varepsilon\to 0} T_\varepsilon. \tag{4.1}$$

However, certain results of the type (4.1) can be deduced now from the boundedness of the solutions (3.4). Using the weak compactness theorem ([28] p. 126) and the Rellich–Kondrachov theorem ([1] p. 143), one can extract a subset $\mathscr{E}_0$ of the interval [0, 1] such that $\inf \mathscr{E}_0 = 0$ and

$$\begin{aligned} &\tilde{u}_\varepsilon \tilde{u} \to {}_0 \quad \text{in } H^1(\Omega; R^n)/N^{\mathrm{ad}} \quad \text{weakly and in} \quad L^2(\Omega; R^n)/N^{\mathrm{ad}} \text{ strongly},\\ &E_\varepsilon \to E_0 \quad \text{in} \quad L^2(\Omega; S^n) \quad \text{weakly},\\ &T_\varepsilon \to T_0 \quad \text{in } L^2(\Omega; S^n) \quad \text{weakly} \quad \text{as} \quad \varepsilon \to 0 \quad \text{in} \quad \mathscr{E}_0. \end{aligned} \tag{4.2}$$

Furthermore, the limit analysis of the equations (3.1) and (3.3) shows that the weak limits $\tilde{u}_0$, E_0, T_0 fulfill the equations

$$\begin{aligned} &E_0 = Du_0 \quad \text{on } \Omega,\\ &\int_\Omega Dv \cdot T_0\, dx = \int_\Omega v \cdot b_0\, dx + \int_{\partial\Omega} v \cdot t\, ds, \qquad \forall v \in V^{\mathrm{ad}} \end{aligned} \tag{4.3}$$

which can be understood as a macroscopic geometric equation and as a macroscopic virtual work principle.

5. Approximation by averaging

In this section we shall present some properties of the method of approximation by averaging which are essential in our approach to the homogenization procedure.

Let us recall that $Y = (y_1, \ldots, y_n)$ denotes the set of n linearly independent vectors in R^n generating the considered family of periodic composites.

Almost everywhere on R^n we define a vector-valued function r by the condition

$$r(x) = \begin{cases} y & \text{if } \exists\ y \in N \quad \text{such that} \quad (x-y) \in C,\\ \text{is undefined otherwise.} \end{cases} \tag{5.1}$$

Let

$$r_\varepsilon(x) = \varepsilon r(x/\varepsilon). \tag{5.2}$$

Obviously, if $\varepsilon > 0$ then r_ε is εY-periodic.

For $f \in L^2(R^n)$ we define its approximation by averaging

$$f^{\varepsilon\,\mathrm{av}}(x) = \varepsilon^{-n}\ \mathrm{meas}^{-1} C \int\limits_{r_\varepsilon(x)+\varepsilon C} f(z)\ dz. \tag{5.3}$$

It is known [4] that such approximation is strongly convergent in $L^2(R^n)$. Consequently, if

$$f_\varepsilon \to f_0 \quad \text{in} \quad L^2(R^n) \quad \text{weakly as} \quad \varepsilon \to 0$$

then

$$f_\varepsilon^{\varepsilon\,\mathrm{av}} \to f_0 \quad \text{in} \quad L^2(R^n) \quad \text{weakly as} \quad \varepsilon \to 0, \tag{5.4}$$

and by virtue of the relations $(4.2)_2$ and $(4.2)_3$ we get

(5.5) and
$$\begin{aligned} &E_\varepsilon^{\varepsilon\,\mathrm{av}} \to E_0 \quad \text{in } L^2(R^n; S^n) \quad \text{weakly} \\ &T_\varepsilon^{\varepsilon\,\mathrm{av}} \to T_0 \quad \text{in } L^2(R^n; S^n) \quad \text{weakly} \quad \text{as} \quad \varepsilon \to 0 \quad \text{in } \mathscr{E}_0. \end{aligned}$$

6. Local fields

We have seen in Sect. 4 that the macroscopic geometric equation and the macroscopic virtual work principle result directly from the formulation (3.1)–(3.3) of the primary problem. It should be mentioned here that the macroscopic constitutive equation does not result directly from the equation (3.2). This is clear because the homogenization procedure homogenizing the material loses its microstructure. To overcome this difficulty, we introduce a set of local coordinate systems in such a way that the microstructure of the material does not vary if it is referred to any of these systems. Mathematical realization of this concept will be obtained by introducing local fields, i.e. functions of two vector variables $(x, y) \in \Omega^* \times R^n (\Omega^* = \Omega + 2C)$ such that

(6.1)
$$\begin{aligned} &u_\varepsilon^{\mathrm{loc}}(x, y) = \varepsilon^{-1} u_\varepsilon\big(r_\varepsilon(x) + \varepsilon y\big), \qquad \text{a.e.} \quad (x, y) \in \Omega^* \times R^n, \\ &\tilde{\tilde{u}}_\varepsilon^{\mathrm{loc}} = u_\varepsilon^{\mathrm{loc}} + L^2(\Omega^*; RD^n), \qquad \text{on } \Omega^* \times R^n, \\ &E_\varepsilon^{\mathrm{loc}}(x, y) = E_\varepsilon\big(r_\varepsilon(x) + \varepsilon y\big) \\ &\text{and} \\ &T_\varepsilon^{\mathrm{loc}}(x, y) = T_\varepsilon\big(r_\varepsilon(x) + \varepsilon y\big), \qquad \text{a.e.} \quad (x, y) \in \Omega^* \times R_n \end{aligned}$$

with

$$RD = \{v;\ R^n \to R^n : Dv = 0\}.$$

The x variable in the condition (5.1) can be understood as a global variable, and the y variable as a local variable centered at $r_\varepsilon(x)$.

In fact, the local fields are deduced from primary fields by a rather simple coordinate transform. Consequently, the local fields must fulfill certain laws of mechanics resulting from the laws (3.1)–(3.3) of the mechanics of primary fields. Indeed, one can show that the following equations involving the local fields hold:

$$E_\varepsilon^{\mathrm{loc}}(x, y) = D\tilde{\tilde{u}}_\varepsilon^{\mathrm{loc}}(x; y)$$

and

$$T_\varepsilon^{\mathrm{loc}}(x, y) = \mathscr{T}\big(y, E_\varepsilon^{\mathrm{loc}}(x, y)\big) \quad \text{if} \quad (r_\varepsilon(x) + \varepsilon y) \in \Omega,$$

$$(6.2) \qquad \int_\Omega \varphi(x) \int_{R^n} Dv(y) \cdot T_\varepsilon^{\mathrm{loc}}(x, y)\,dy\,dx = \varepsilon^{1-n} \int_\Omega \varphi(x) \int_\Omega v\big((z - r_\varepsilon(x))/\varepsilon\big) \cdot b_\varepsilon(z)\,dz\,dx$$

for $\varphi \in C_0^\infty(\Omega)$ and $v \in C_0^\infty(R^n, R^n)$ such that $(\operatorname{supp}\varphi + \varepsilon C + \varepsilon \operatorname{supp} v) \subset \Omega$.

Equations $(6.2)_1$–$(6.2)_3$ can be understood as local forms of geometric equations, constitutive equations and of virtual work principle. However, the local fields have other properties which will prove to be very useful in our analysis.

First, let us observe that the εY-periodicity of r_ε implies the result

$$(6.3) \text{ and } \quad \begin{aligned} E_\varepsilon^{\mathrm{loc}}(x, y+z) &= E_\varepsilon^{\mathrm{loc}}(x + \varepsilon z, y) \\ T_\varepsilon^{\mathrm{loc}}(x, y+z) &= T_\varepsilon^{\mathrm{loc}}(x + \varepsilon z, y), \quad \text{a.e.} \quad (x, y) \in \Omega^* \times R^n \quad \forall z \in N \end{aligned}$$

which can be called quasi-periodicity of local deformations and of local stresses. Next, from Eq. (5.3) we get

$$(6.4) \text{ and } \quad \begin{aligned} E_\varepsilon^{\varepsilon\,\mathrm{av}}(x) &= \operatorname{meas}^{-1} C \int_C E_\varepsilon^{\mathrm{loc}}(x, y)\,dy \\ T_\varepsilon^{\varepsilon\,\mathrm{av}}(x) &= \operatorname{meas}^{-1} C \int_C T_\varepsilon^{\mathrm{loc}}(x, y)\,dy, \quad \text{a.e.} \quad x \in \Omega^*, \end{aligned}$$

what indicates equality of appropriate mean values of primary fields and local fields.

Furthermore, the concept of zero extension of E_ε and T_ε outside Ω implies

$$(6.5) \qquad E_\varepsilon^{\mathrm{loc}}(x, y) = T_\varepsilon^{\mathrm{loc}}(x, y) = 0 \quad \text{if} \quad (r_\varepsilon(x) + \varepsilon y) \notin \Omega.$$

Finally, note that as a result of the integral calculus we have

$$\int_{\Omega^*} \int_C E_\varepsilon^{\mathrm{loc}}(x, y) \cdot T_\varepsilon^{\mathrm{loc}}(x, y)\,dy\,dx = \operatorname{meas} C \int_\Omega E_\varepsilon \cdot T_\varepsilon\,dx$$

and by the relation (3.3) we get the following energetic identity for local fields:

$$(6.6) \qquad \int_\Omega \int_C E_\varepsilon^{\mathrm{loc}}(x, y) \cdot T_\varepsilon^{\mathrm{loc}}(x, y)\,dy\,dx = \operatorname{meas} C \Big\{ \int_\Omega [D\tilde{u} \cdot T_\varepsilon + (\tilde{u}_\varepsilon - \tilde{u}) \cdot b_\varepsilon]\,dx + \int_{\partial\Omega} (\tilde{u}_\varepsilon - \tilde{u}) \cdot t\,ds \Big\}$$

with a certain arbitrary $\tilde{u} \in \tilde{U}^{\mathrm{ad}}$. Consequently, using the standard methods of "a priori" estimates, we obtain the following, local in the second variable, estimates for local fields:

$$(6.7) \qquad \max\big(\|\tilde{\tilde{u}}_\varepsilon^{\mathrm{loc}} : L^2(\Omega^*; H^1(\omega; R^n)/RD^n)\|, \|E_\varepsilon^{\mathrm{loc}}\|_{\Omega^* \times \omega}, \|T^{\mathrm{loc}}\|_{\Omega^* \times \omega}\big) \leqslant c(\omega)$$

for all open bounded ω in R^n with $c(\omega)$ independent of ε.

7. Microscopic fields

Formally we define the microscopic fields as limits of local fields, i.e.

$$\tilde{\tilde{u}}_\mu = \lim_{\varepsilon\to 0} \tilde{\tilde{u}}_\varepsilon^{\text{loc}}, \qquad E_\mu = \lim_{\varepsilon\to 0} E_\varepsilon^{\text{loc}}, \qquad T_\mu = \lim_{\varepsilon\to 0} T_\varepsilon^{\text{loc}} \tag{7.1}$$

without precise specification of the topology of convergence. As in Sect. 4 partial results concerning the problem can be deduced from boundedness (6.7). One can prove that there exist

$$\tilde{\tilde{u}}_0^{\text{loc}} \in L^2\big(\Omega^*;\, H^1_{\text{loc}}(R^n;\, R^n)/RD^n\big),$$
$$E_0^{\text{loc}} \in L^2\big(\Omega^*;\, L^2_{\text{loc}}(R^n;\, S^n)\big) \quad \text{and}$$
$$T_0^{\text{loc}} \in L^2\big(\Omega^*;\, L^2_{\text{loc}}(R^n;\, S^n)\big)$$

such that for all open bounded $\omega \subset R^n$ there exists a subset $\mathscr{E}_\omega$ of $\mathscr{E}_0$ such that $\inf \mathscr{E}_\omega = 0$ and

$$\begin{aligned} &(\tilde{\tilde{u}}_\varepsilon^{\text{loc}} - u_0^{\text{loc}})|_{\Omega^*\times\omega} \to 0 \quad \text{in } L^2(\Omega^*;\, H^1(\omega;\, R^n)/RD^n) \quad \text{weakly},\\ &(E_\varepsilon^{\text{loc}} - E_0^{\text{loc}})|_{\Omega^*\times\omega} \to 0 \quad \text{in } L^2(\Omega^*\times\omega;\, S^n) \quad \text{weakly},\\ &(T_\varepsilon^{\text{loc}} - T_0^{\text{loc}})|_{\Omega^*\times\omega} \to 0 \quad \text{in } L^2(\Omega\times\omega;\, S^n) \quad \text{weakly as} \quad \varepsilon\to 0 \quad \text{in } \mathscr{E}_\omega. \end{aligned} \tag{7.2}$$

Now we should proceed to the limit analysis of Eqs. (6.2)–(6.6) characterizing the mechanics of local fields. We claim that the result of such analysis is as follows:

$$E_0^{\text{loc}} = D\tilde{\tilde{u}}_0^{\text{loc}} \quad \text{on} \quad \Omega\times R^n, \tag{7.3}$$

$$T_0^{\text{loc}}(x, y) = \mathscr{T}\big(y, E_0^{\text{loc}}(x, y)\big), \quad \text{a.e.} \quad (x, y)\in\Omega^*\times R^n, \tag{7.4}$$

$$\int_{R^n} Dv(y)\cdot T_0^{\text{loc}}(x, y)\,dy = 0, \quad \forall v\in C_0^\infty(R^n;\, R^n), \quad \text{a.e.} \quad x\in\Omega, \tag{7.5}$$

$$E_0^{\text{loc}}(x, y+z) = E_0^{\text{loc}}(x, y) \tag{7.6}$$

and

$$T_0^{\text{loc}}(x, y+z) = T_0^{\text{loc}}(x, y), \quad \forall z\in N, \quad \text{a.e.} \quad (x, y)\in\Omega^*\times R^n, \tag{7.7}$$

$$E_0(x) = \text{meas}^{-1} C \int_C E_0^{\text{loc}}(x, y)\,dy \tag{7.8}$$

and

$$T_0(x) = \text{meas}^{-1} C \int_C T_0^{\text{loc}}(x, y)\,dy, \quad \text{a.e.} \quad x\in\Omega, \tag{7.9}$$

$$u_0^{\text{loc}} = E_0^{\text{loc}} = T_0^{\text{loc}} = 0, \quad \text{on} \quad (\Omega^*\setminus\Omega)\times R^n, \tag{7.10}$$

$$\lim_{\varepsilon\to 0} \int_{\Omega^*}\int_C E_\varepsilon^{\text{loc}}(x, y).\, T_\varepsilon^{\text{loc}}(x, y)\,dy\,dx = \text{meas}\, C \int_\Omega E_0\cdot T_0\,dx. \tag{7.11}$$

Equations (7.3) and (7.6)–(7.11) follow directly from the previous results. The remaining ones require some additional analysis.

To prove Eq. (7.5), it is sufficient to show that the right hand side of the relation $(6.2)_3$ tends to zero. Note that by virtue of the Hölder inequality we have

$$\int_{\Omega} |v((z-r_\varepsilon(x))/\varepsilon)\cdot b_\varepsilon(z)|\,dz \leqslant \Big\{\int_{\Omega} |v((z-r_\varepsilon(x))/\varepsilon)|^{p'}dz\Big\}^{1/p'} \|b_\varepsilon;\, L^p(\Omega;\, R^n)\|.$$

But the coordinate transform

$$y = (z-r_\varepsilon(x))/\varepsilon$$

gives

$$\int_{\Omega} |v((z-r_\varepsilon(x))/\varepsilon)|^{p'}dz = \varepsilon^n \int_{\varepsilon^{-1}(\Omega-r_\varepsilon(x))} |v(y)|^{p'}dz \leqslant \varepsilon^n \|v;\, L^{p'}(R^n;\, R^n)\|^{p'}.$$

Hence the right hand side of the relation $(6.2)_3$ can be estimated by

$$\varepsilon^{p-n/p}\|\varphi;\, L^1(\Omega)\|\;\|v;\, L^{p'}(R^n;\, R^n)\|\;\|b_\varepsilon;\, L^p(\Omega;\, R^n)\|$$

and the condition (7.5) follows because by Assumption A7 the sequence b_ε is bounded in L^p and $p > n$.

We shall prove Eq. (7.4) in several steps. Let us note that the Y-periodicity of $T_0^{\mathrm{loc}}(x, y)$ in y makes the condition (7.5) equivalent to

$$(7.12)\qquad \int_C Dv(y)\cdot T_0^{\mathrm{loc}}(x, y)\,dy = 0, \quad \text{for all} \quad Y\ \text{periodic}$$

$$v \in H^1_{\mathrm{loc}}(R^n;\, R^n), \quad \text{a.e.} \quad x \in \Omega.$$

Next, let us choose a function $\mathscr{X} \in L^2(\Omega;\, H^1_{\mathrm{loc}}(R^n;\, R^n))$ such that

$$(7.13)\qquad E_0^{\mathrm{loc}}(x, y) = E_0(x) + D\mathscr{X}(x, y), \quad \text{a.e.} \quad (x, y) \in \Omega\times R^n.$$

Such a function exists by virtue of Duvaut's lemma ([20] p. 24 and [14]) and the conditions (7.6) and (7.8). Taking the function $\mathscr{X}$ as a test function in the relation (7.12), we get

$$\int_C E_0^{\mathrm{loc}}(x, y)\cdot T_0^{\mathrm{loc}}(x, y)\,dy = \int_C E_0(x)\cdot T_0^{\mathrm{loc}}(x, y)\,dy = \operatorname{meas} C E_0(x)\cdot T_0(x).$$

Consequently, Eq. (7.11) leads to the identity

$$(7.14)\qquad \lim_{\varepsilon\to 0}\int_{\Omega^*}\int_C E_\varepsilon^{\mathrm{loc}}(x, y)\cdot T_\varepsilon^{\mathrm{loc}}(x, y)\,dy\,dx = \int_{\Omega}\int_C E_0^{\mathrm{loc}}(x, y)\cdot T_0^{\mathrm{loc}}(x, y)\,dy\,dx$$

which enables us to deduce Eq. (7.4) from Eq. $(6.2)_2$ using the standard arguments [21] of limit analysis of monotonic constitutive functions.

8. Self-equilibrated periodic deformation state

Let us analyse the set of equations (7.3)–(7.8). It is clear that taking the x variable as a parameter one can consider this set of equations as a variational formulation of a one-parameter family of self-equilibrium problems of infinite Y-periodic composite in Y-periodic deformation state with mean deformation prescribed by Eq. (7.8). For our aims it is convenient to formulate these problems in the following, somehow moregener al form:

P2. For a given $F \in S^n$ find

$$\tilde{\tilde{u}}_F \in H^1_{\mathrm{loc}}(R^n; R^n)/RD^n, \quad E_F \in L^2_{\mathrm{loc}}(R^n; S^n) \quad \text{and} \quad T_F \in L^2_{\mathrm{loc}}(R^n; S^n)$$

such that

$$E_F = D\tilde{\tilde{u}}_F \quad \text{on } R^n, \tag{8.1}$$

$$T_F(y) = \mathcal{T}\big(y, E_F(y)\big), \quad \text{a.e.} \quad y \in R^n, \tag{8.2}$$

$$\int_{R^n} Dv \cdot T_F dy = 0, \quad \forall v \in C_0^\infty(R^n; R^n), \tag{8.3}$$

$$\int_C E_F dy = \operatorname{meas} CF, \tag{8.4}$$

$$E_F \quad \text{is} \quad Y\text{-periodic.} \tag{8.5}$$

As in the previous section we can represent the Y-periodic deformation field E_F in the form

$$E_F = F + D\mathcal{X}_F \quad \text{on} \quad R^n \tag{8.6}$$

with a certain Y-periodic $\mathcal{X}_F \in H^1_{\mathrm{loc}}(R^n; R^n)$ such that

$$\int_C \mathcal{X}_F dy = 0. \tag{8.7}$$

Using again the arguments of Y-periodicity of stress fields, we can formulate the virtual work principle (8.3) in the form

$$\int_C Dv \cdot T_F dy = 0, \quad \text{for all } Y\text{-periodic} \quad v \in H^1_{\mathrm{loc}}(R^n; R^n). \tag{8.8}$$

Consequently, the problem P2 has the following equivalent formulation:

P3. For a given $F \in S^n$ find:

$$\mathcal{X}_F \in H^1_{\mathrm{loc}}(R^n; R^n), \quad E_F \in L^2_{\mathrm{loc}}(R^n; S^n) \quad \text{and} \quad T_F \in L^2_{\mathrm{loc}}(R^n; S^n)$$

such that Eqs. (8.2) and (8.6)–(8.8) are fulfilled.

As in the case of the problem $\mathrm{P1}_\varepsilon$, the existence and the uniqueness can easily be proved. Moreover, it can be shown that the solution is Lipschitz continuous in F, i.e. there exists a positive constant c such that

$$\max(\|\mathcal{X}_{F_2} - \mathcal{X}_{F_1}\|_{C;1}, \|E_{F_2} - E_{F_1}\|_C, \|T_{F_2} - T_{F_1}\|_C) \leqslant c|F_2 - F_1|, \quad \forall F_1, F_2 \in S^n. \tag{8.9}$$

9. Effective constitutive function

The existence and the uniqueness of solutions for the problem P3 enable us to define the effective constitutive function $\mathcal{T}^{\mathrm{eff}}; S^n \to S^n$ putting

$$\mathcal{T}^{\mathrm{eff}}(F) = \operatorname{meas}^{-1} C \int_C T_F(y)\, dy. \tag{9.1}$$

The Lipschitz continuity of T_F in F implies Lipschitz continuity of the effective constitutive function, i.e. $\exists c > 0$ such that

(9.2) $$|\mathcal{T}^{\mathrm{eff}}(F_2)-\mathcal{T}^{\mathrm{eff}}(F_1)| \leqslant c|F_2-F_1|, \quad \forall F_1, F_2 \in S^n.$$

Moreover, the assumed monotony of $\mathcal{T}(y, E)$ in E implies monotony of $\mathcal{T}^{\mathrm{eff}}$, i.e. $\exists c > 0$ such that

(9.3) $$(F_2-F_1)\cdot[\mathcal{T}^{\mathrm{eff}}(F_2)-\mathcal{T}^{-\mathrm{eff}}(F_1)] \geqslant c|F_2-F_1|^2, \quad \forall F_1, F_2 \in S^n.$$

10. Homogenized problem

It results from Sects. 8 and 9 that the set of equations (7.3)–(7.9) implies

(10.1) $$T_0(x) = \mathcal{T}^{\mathrm{eff}}(E_0(x)), \quad \text{a.e.} \quad x \in \Omega.$$

Thus we see that the weak limits $\tilde{u}_0$, E_0, T_0 constitute a solution for the following equilibrium problem:

P4. For given $b_0 \in L^2(\Omega; R^n)$ and $t \in L^2(\partial\Omega; R^n)$ find:

$$\tilde{u}_0 \in \tilde{U}^{\mathrm{ad}}, \quad E_0 \in L^2(\Omega; S^n) \quad \text{and} \quad T_0 \in L^2(\Omega; S^n)$$

such that Eqs. (4.3) and (10.1) are satisfied.

The problem P4 can be proved to have unique solutions.

11. Homogenization theorems

The uniqueness of solutions for P4 implies that the convergences (4.2) hold with $\mathcal{E}_0 =]0, 1]$. Hence we have proved the following result:

THEOREM 1. *Suppose that the assumptions* A1–A7 *are fulfilled. Then*

$$u_\varepsilon \to u_0 \quad \text{in} \quad H^1(\Omega; R^n)/N^{\mathrm{ad}} \text{ weakly and} \quad \text{in } L^2(\Omega; R^n)/N^{\mathrm{ad}} \text{ strongly},$$
$$E_\varepsilon \to E_0 \quad \text{in } L^2(\Omega; S^n) \text{ weakly and}$$
$$T_\varepsilon \to T_0 \quad \text{in } L^2(\Omega; S^n) \text{ weakly}$$

as $\varepsilon \to 0$ *with* $(u_\varepsilon, E_\varepsilon, T_\varepsilon)$ *and* (u_0, E_0, T_0) *denoting solutions for the problems* P_ε^1 *and* P4, *respectively* ■

Next, the uniqueness of solutions for P3 implies that the convergences (7.2) hold with $\mathcal{E}_\omega =]0, 1]$. Moreover, using the classical energy method, we can prove that these convergences are locally strong, i.e.

THEOREM 2. *Suppose that the assumptions* A1–A7 *are fulfilled. Then for each open bounded* ω *in* R^n

$$(\tilde{u}_\varepsilon^{\mathrm{loc}}-\tilde{u}_0^{\mathrm{loc}})|_{\Omega^*\times\omega} \to 0 \quad \text{in} \quad L^2(\Omega^*; H^1(\omega; R^n)/RD^n) \text{ strongly},$$
$$(E_\varepsilon^{\mathrm{loc}}-E_0^{\mathrm{loc}})|_{\Omega^*\times\omega} \to 0 \quad \text{in} \quad L^2(\Omega^*\times\omega; S^n) \text{ strongly and}$$
$$(T_\varepsilon^{\mathrm{loc}}-T_0^{\mathrm{loc}})|_{\Omega^*\times\omega} \to 0 \quad \text{in} \quad L^2(\Omega^*\times\omega; S^n) \text{ strongly}$$

as $\varepsilon \to 0$ *with* $u_\varepsilon^{\mathrm{loc}}$, $E_\varepsilon^{\mathrm{loc}}$, $T_\varepsilon^{\mathrm{loc}}$ *defined by* (6.1) *and*

$$u_0^{\mathrm{loc}}(x, y) = E_0(x)y+\mathcal{X}_{E_0(x)}(y), \quad \text{a.e.} \quad (x, y) \in \Omega\times R^n,$$

$$E_0^{\mathrm{loc}}(x, y) = E_{E_0(x)}(y), \qquad \text{a.e.} \quad (x, y) \in \Omega \times R^n,$$
$$T_0^{\mathrm{loc}}(x, y) = T_{E_0(x)}(y), \qquad \text{a.e.} \quad (x, y) \in \Omega \times R^n,$$
$$\tilde{\tilde{u}}_0^{\mathrm{loc}} = E_0^{\mathrm{loc}} = T_0^{\mathrm{loc}} = 0 \qquad \text{on} \quad (\Omega^* \setminus \Omega) \times R^n,$$

where $(\mathscr{X}_{E_0(x)}, E_{E_0(x)}, T_{E_0(x)})$ *denotes the solution for* P3 *with* $F = E_0(x)$ ■

Theorems 1 and 2 justify our definitions of the macroscopic (4.1) and the microscopic (7.1) fields. Obviously,

$$\begin{aligned} &\tilde{u}_m = \tilde{u}_0, \quad E_m = E_0, \quad T_m = T_0, \quad \text{on } \Omega,\\ &\tilde{\tilde{u}}_\mu = \tilde{\tilde{u}}_0^{\mathrm{loc}}, \quad E_\mu = E_0^{\mathrm{loc}}, \quad T_\mu = T_0^{\mathrm{loc}}, \quad \text{on } \Omega \times R^n. \end{aligned} \tag{11.1}$$

Furthermore, Theorem 2 provides us with a strongly convergent approximation for local fields. Consequently, taking into account the clear correspondence between primary and local fields, it seems that Theorem 2 should give us an opportunity for constructing strongly convergent approximations for primary fields.

THEOREM 3. *Suppose that the assumptions* A1–A7 *are fulfilled. Then*

$$(E_\varepsilon - E_\varepsilon^{\mathrm{app}}) \to 0 \quad \text{in } L^2(\Omega; S^n) \quad \text{strongly},$$
$$(T_\varepsilon - T_\varepsilon^{\mathrm{app}}) \to 0 \quad \text{in } L^2(\Omega; S^n) \quad \text{strongly} \quad \text{as} \quad \varepsilon \to 0,$$
with
$$E_\varepsilon^{\mathrm{app}}(x) = E^p(E_0^{\varepsilon\,\mathrm{av}}(x), x/\varepsilon), \quad \text{on } \Omega,$$
$$T_\varepsilon^{\mathrm{app}}(x) = T^p\big(E_0^{\varepsilon\,\mathrm{av}}(x), x/\varepsilon\big), \quad \text{on } \Omega$$

where

$$E^p(F, y) = E_F(y), \qquad T^p(F, y) = T_F(y) \;■$$

The proofs of the considered two convergences are similar. Therefore we shall analyse only the first one.

Applying the partition

$$\bigcup_{\xi \in \varepsilon N} (\xi + \varepsilon C) \cap \Omega$$

of Ω and using the coordinate transform

$$x = \xi + \varepsilon y$$

on each set $(\xi + \varepsilon C) \cap \Omega$, we get

$$\|E_\varepsilon - E_\varepsilon^{\mathrm{app}}\|_\Omega^2 = \varepsilon^n \sum_{\xi \in \varepsilon N} \int_{C \cap [(\Omega - \xi)/\varepsilon]} |E_\varepsilon^{\mathrm{loc}}(\xi, y) - E^p(E_0^{\varepsilon\,\mathrm{av}}(\xi), y)|^2 dy$$
$$\leqslant \mathrm{meas}^{-1} C \|E_\varepsilon^{\mathrm{loc}} - E \circ E_0^{\varepsilon\,\mathrm{av}}\|^2_{(\Omega + 2\varepsilon C) \times C}.$$

Next, by the triangle inequality we have

$$\begin{aligned} \|E_\varepsilon^{\mathrm{loc}} - E^p \circ E_0^{\mathrm{av}}\|_{(\Omega+2\varepsilon C)\times C} \leqslant{}& \|E_\varepsilon^{\mathrm{loc}} - E_0^{\mathrm{loc}}\|_{(\Omega+2\varepsilon C)\times C}\\ &+ \|E_0^{\mathrm{loc}} - E^p \circ E_0\|_{(\Omega+2\varepsilon C)\times C} + \|E^p \circ E_0 - E^p \circ E_0^{\varepsilon\,\mathrm{av}}\|_{(\Omega+2\varepsilon C)\times C}. \end{aligned} \tag{11.2}$$

The first term of the right hand side of the relation (11.2) converges to zero by virtue of Theorem 2. For the second term we have

$$\|E_0^{\mathrm{loc}} - E^p \circ E_0\|^2_{(\Omega+2\varepsilon C)\times C} = \|E^p(0, \cdot)\|^2_{[(\Omega+2\varepsilon C)\setminus\Omega]\times C}$$
$$= \mathrm{meas}[(\Omega + 2\varepsilon C)\setminus\Omega] \int_C |E^p(0, y)|^2 dy.$$

Consequently, the second term converges to zero, because the Lebesgue measure of $[(\Omega+2\varepsilon C)\setminus\Omega]$ tends to zero by virtue of properties of domains with finite perimeter [15].

The estimate (8.9) implies

$$||E^p \circ E_0 - E^p \circ E_0^{\varepsilon\, av}||_{(\Omega+2\varepsilon C)\times C} \leqslant c\, ||E_0 - E_0^{\varepsilon\, av}||_{\Omega+2\varepsilon C}$$

and the last term converges to zero by the properties of approximation by averaging [4].

Note. The paper was supported by the Polish Academy of Sciences within the Program I-23.

References

1. R. A. ADAMS, *Sobolev spaces*, Academic Press, New York 1975.
2. A. AMBROSETTI, C. SBORDONE, *F-convergenza e G-convergenza per problemi non lineari di tipo ellittico*, Boll. U.M.I., **13A**, 352–362, 1976.
3. M. ARTOLA, G. DUVAUT, *Un resultat d'homogeneisation pour une classe de problemes de diffusions non-lineaires stationnaires*, Ann. Fac. Sc. Toulouse, Ser. 5, **4**, 1–29, 1982.
4. J. P. AUBIN, *Approximation for elliptic boundary value problems*, Wiley Int., New York 1972.
5. A. BENSOUSSAN, J. L. LIONS, G. PAPANICOLAOU, *Asymptotic analysis for periodic structures*, North Holland, Amsterdam 1978.
6. L. BOCCARDO, F. MURAT, *Remarques sur l'homogeneisation de certains problemes quasi-lineaires*, Port. Math., **41**, 535–562, 1982.
7. L. BOCCARDO, F. MURAT, *Homogeneisation de problemes quasi-lineaires*, Proc. Meet., Studio di Problemi Limite della Analisi Funzionale, Bressanone 1981, Ed. Pitagora Bologna, pp. 13–51, 1982.
8. F. E. BROWDER, *Pseudo-monotone operators and nonlinear elliptic boundary value problems on unbounded domains*, Proc. Nat. Acad. Sci. USA, **74**, 2659–2661, 1977.
9. E. DE GIORGI, T. FRANZONI, *Una presentazione sintetica dei limiti generalizzati*, Port. Math., **41**, 405–436, 1982.
10. H. DUMONTET, *Homogeneisation des materiaux stratifies de type elastique lineaire, non-lineaire, visco-elastique*, These, Universite Pierre et Marie Curie, Paris 1983.
11. G. DUVAUT, J. L. LIONS, *Les inequations en mecanique et en physique*, Dunod, Paris 1972.
12. G. DUVAUT, *Homogeneisation dune plaque a struture periodique en theorie non-lineaire de von Kàrmàn*, C.R. Journees d'Analyse Non Lineaire, Besançon 1977.
13. G. DUVAUT, *Cours sur les methodes variationnelles et la dualite*, in: Duality and Complementarity in Mechanics of Solids, pp. 173–273, Ossolineum, Warszawa 1979.
14. G. DUVAUT, *Methodes variationnelles. Initiation et application*, Cours du D.E.A. d'Hydrothermodynamique, E.N.S.T.A. 1979.
15. H. FEDERER, *Geometric measure theory*, Springer V., Berlin 1969.
16. G. FICHERA, *Existence theorems in elasticity*, in: Handbuch der Physik, Band VI a/2, Springer V., Berlin 1972.
17. N. FUSCO, G. MOSCARIELLO, *On the homogenization of quasi-linear divergence structure operators*, Univ. Studi Napoli, Dip. Mat. Appl., R. Caccioppoli, 55, 1984.
18. J. GOBERT, *Une inegalité fondamentale de la theorie de l'elasticité*, Bull. Soc. Roy. Sci., Liege, 31, 3–4, 182–192, 1962.
19. R. KLUGE, *Nichtlineare Variationsungleichungen und Extremalaufgaben*, Veb Deutscher V., Berlin 1979.
20. F. LENE, *Contribution a l'etude des materiaux composites et de leur endommagement*, These de Doctorat d'Etat, Universite Pierre et Marie Curie, Paris 1984.
21. J. L. LIONS, *Quelques methodes de resolution des problemes aux limites non lineaires*, Dunod, Paris 1969.

22. P. Marcellini, *Periodic solutions and homogenization of nonlinear variational problems*, Ann. Mat. Pura Appl., **4**, 111, 139–152, 1978.
23. P. Marcellini, C. Sbordone, *Sur quelques questions de G-convergence et d'homogeneisation non-lineaire*, C.R.A.S. 284, 535–537, Paris 1977.
24. U. E. Raitum, *On the G-convergence of quasi-linear elliptic operators with unbounded coefficients*, Sov. Met. Dokl., **24**, 472–475, 1981 [in Russian].
25. E. Sanchez Palencia, *Nonhomogeneous media and vibration theory*, Springer V., Berlin 1980.
26. P. Suquet, *Plasticite et homogeneisation*, These de Doctorat d'Etat, Universite Pierre et Marie Curie, Paris 1982.
27. L. Tartar, *Nonlinear constitutive relations and homogenization*, in: Contemporary Developments, Ed. La Penha, North Holland. Amsterdam 1978.
28. K. Yosida, *Functional analysis*, Springer V., Berlin 1980.

INSTITUTE OF CONSTRUCTION ENGINEERING
TECHNICAL UNIVERSITY OF ŁÓDŹ.

Received September 27, 1984.