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ADDITIONS TO THE ARTICLES*, "ON A NEW CLASS OF THEOREMS," AND "ON PASCAL'S THEOREM."

[*Philosophical Magazine*, XXXVII. (1850), pp. 363—370.]

First addition.—I have alluded in the second of the above articles to a more general theorem, comprising, as a particular case, the theorem there given for the simultaneous evanescence of two quadratic functions of $2n$ letters, on n linear equations becoming instituted between the letters.

In order to make this generalization intelligible, I must premise a few words on the Theory of Orders, a term which I have invented with particular reference to quadratic functions, although obviously admitting of a more extended application. A linear function of all the letters entering into a function or system of functions under consideration I call an order of the letters, or simply an order. Now it is clear that we may always consider a function of any number of letters as a function of as many orders as there are letters; but in certain cases a function may be expressed in terms of a fewer number of orders than it has letters, as when the general characteristic function of a conic becomes that of a pair of crossing lines or a pair of coincident lines, in which event it loses respectively one and two orders, and so for the characteristic of a conoid becoming that of a cone, a pair of planes or two coincident planes, in which several events, a function of four letters becomes that of only three orders, or two orders, or one order, respectively. When a function may be expressed by means of r orders less than it contains letters, I call it a function minus r orders. I now proceed to state my theorem.

Let U and V be functions each of the same m letters, and suppose that the determinant in respect of those letters of $U + \mu V$ contains i pairs of

[* pp. 138, 139 above. Ed.]

equal linear factors of μ; then it is possible, by means of i linear equations instituted between the letters, to make U and V each become functions of *the same* $m-2i$ orders; and conversely, if by i equations between the letters U and V may be made functions of *the same* $m-2i$ orders, the determinant of $U+\mu V$ considered as a function of μ will contain i square factors.

Thus when $m=2n$ and $i=n$, U and V will each become functions of zero orders, that is, will both disappear, provided that on the institution of a certain system of n linear equations, among the letters of which U and V are functions, the determinant of $(U+\mu V)$ is a perfect square,—which is the theorem given in the article referred to.

So for example if U and V be quadratic functions of four letters, and therefore the characteristics of two conoids, $\square\,(U+\mu V)$ being a perfect square, expresses that these conoids have a straight line in common lying upon each of their surfaces.

If U and V be quadratic functions of three letters only, and admit therefore of being considered as the characteristics of two conics, $\square\,(U+\mu V)$ containing a square factor, is indicative of these conics having a common tangent at a common point, that is, of their touching each other at some point; for it is easily shown that the disappearance of two orders from any quadratic function by virtue of one linear function of its letters being zero, indicates that the line, plane, &c. of which the linear function is the characteristic is a tangent to the curve, surface, &c. of which the quadratic function is the characteristic.

I pass now to a generalization of the theorem which shows how to express, under the form of a double determinant, the resultant of one linear and two quadratic homogeneous functions of three letters (which I should have given in the original paper, had I not there been more intent upon developing an ascending scale than of expatiating upon a superficial ramification of analogies), and which constitutes my *Second addition* to that paper, to wit—

If U and V be homogeneous quadratic, and $L_1, L_2 \dots L_n$ homogeneous linear functions of $(n+2)$ letters $x_1, x_2 \dots x_{n+2}$, the determinant of the entire system of $n+2$ functions is equal to

$$\underset{\lambda,\,\mu}{\square}\ \underset{x_1,\,x_2\,\dots\,x_{n+2}\,t_1,\,t_2\,\dots\,t_n}{\boxed{}}\ \{\lambda U+\mu V+L_1t_1+L_2t_2+\dots+L_nt_n\};$$

the demonstration is precisely similar to the analytical one given in the September Number* for the particular case of $n=1$.

When $n=0$, we revert to Mr Boole's theorem of elimination between U and V already adverted to. The proof, it will be easily recognized, does not require the application of the more general theorem relative to the simul-

[* p. 140 above.]

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taneous depression of orders of two quadratic functions, but only the limited one before given, which supplies the conditions of their simultaneous disparition. I now proceed to develope more particularly certain analogies between the theory of the mutual contacts of two conics, and that of the tangencies to the intersection of two conoids.

But here again I must anticipate some of the results which will be given in my forthcoming memoir on Determinants and Quadratic Functions, by explaining what is to be understood by minor determinants, and the relation in which they stand to the complete determinant in which they are included. This preliminary explanation, and the statement of the analogies above alluded to, will constitute my *Third* and last *addition.*

Imagine any determinant set out under the form of a square array of terms. This square may be considered as divisible into lines and columns. Now conceive any one line and any one column to be struck out, we get in this way a square, one term less in breadth and depth than the original square; and by varying in every possible manner the selection of the line and column excluded, we obtain, supposing the original square to consist of n lines and n columns, n^2 such minor squares, each of which will represent what I term a First Minor Determinant relative to the principal or complete determinant. Now suppose two lines and two columns struck out from the original square, we shall obtain a system of $\left\{\frac{n(n-1)}{2}\right\}^2$ squares, each two terms lower than the principal square, and representing a determinant of one lower order than those above referred to. These constitute what I term a system of Second Minor Determinants; and so in general we can form a system of rth minor determinants by the exclusion of r lines and r columns, and such system *in general* will contain

$$\left\{\frac{n(n-1)\dots(n-r+1)}{1.2\dots r}\right\}^2$$

distinct determinants.

I say "*in general*"; because if the principal determinant be totally or partially symmetrical in respect to either or each of its diagonals, the number of distinct determinants appertaining to each system of minors will undergo a material diminution, which is easily calculable.

Now I have established the following law:—

The whole of a system of rth minors being zero, implies only $(r+1)^2$ equations, that is, by making $(r+1)^2$ of these minors zero, all will become zero; and this is true, no matter what may be the dimensions or form of the complete determinant. But furthermore, if the complete determinant be formed from a quadratic function, so as to be symmetrical about one of its diagonals, then $\frac{1}{2}(r+1)(r+2)$ only of the rth minors being zero, will serve

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to imply that all these minors are zero. Of course, in applying these theorems, care must be taken that the $(r+1)^2$ or $\frac{1}{2}(r+1)(r+2)$ selected equations must be mutually non-implicative, and shall constitute independent conditions.

In the application I am about to make of these principles, we shall have only to deal with a system of *first* minors and of a *symmetrical* determinant. If three of these properly selected be zero, from the foregoing it appears that all must be zero.

Now let U and V be characteristics of two conics, that is, let each be a function of only three letters, it may be shown (see my paper* in the *Cambridge and Dublin Mathematical Journal* for November, 1850) that the different species of contacts between these two conics will correspond to peculiar properties of the compound characteristic $U + \mu V$.

If the determinant of this function have two equal roots, the conics simply touch; if it have three equal roots, the conics have a single contact of a higher order, that is, the same curvature; if its six first minors become zero simultaneously for the same value of μ, the conics have a double contact. If the same value of μ, which makes all these first minors zero, be at the same time not merely a double root (as of analytical necessity it always must be) but a treble root of

$$\square (U + \mu V) = 0,$$

then the conics have a single contact of the highest possible order short of absolute coincidence, that is, they meet in four consecutive points.

The parallelism between this theory and that of two quadratic functions P, Q, and one linear function L† of four letters, say x, y, z, t, is exact‡. For let $P + Lu + \mu Q$ be now taken as our compound characteristic (a function, it will be observed, of five letters, x, y, z, t, u); if its determinant have two equal roots, L has two consecutive points in common with the intersection of P and Q, that is, passes through a tangent to that intersection; if it have three equal roots, L has three consecutive points in common with the said intersection, that is, is an osculating plane thereto; if its fifteen first minors admit of all being made simultaneously zero, L has a double contact with the intersection of P and Q, that is, it is a tangent plane to some one of the four cones of the second order containing this intersection;

[* p. 119 above.]

† Observe that $P=0$, $Q=0$, $L=0$ now express the equations to two conoids and a plane respectively.

‡ This parallelism may be easily shown analytically to imply, and be implied, in the geometrical fact, that the contact of the plane L with the intersection of the two surfaces P and Q, is of exactly the same kind as the contact (which must exist) between the two conics which are the intersections of P and Q respectively with the plane L.

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if the same linear function of μ which enters into all these first minors be contained cubically in the complete determinant, then the plane L passes through four consecutive points of the intersection of P and Q, and the points where it meets the curve will be points of contrary plane flexure; and, as it seems to me, at such points the tangential direction of the curve must point to the summit of one or other of the four cones above alluded to*. In assigning the conditions for L being a double tangent plane to the intersection of P and Q, we may take any three independent minors at pleasure equal to zero. One of these may be selected so as to be clear of the coefficients of L; in fact, the determinant of $P+\mu Q$ will be a first minor of $P+\mu Q+Lu$; μ may thus be determined by a biquadratic equation; and then, by properly selecting the two other minors, we may obtain two equations in which only the first powers of the coefficients of x, y, z, t in L appear, and may consequently obtain L under the form of

$$(ae+\alpha)\,x+(be+\beta)\,y+(ce+\gamma)\,z+(de+\delta)\,t,$$

where a, α; b, β; c, γ; d, δ will be known functions of any one of the four values of μ. The point of contact being given will then serve to determine e, and we shall thus have the equation to each of the four double tangent planes at any given point fully determined.

In the foregoing discussions I have freely employed the word *characteristic* without previously defining its meaning, trusting to that being apparent from the mode of its use. It is a term of exceeding value for its significance and brevity. The characteristic of a geometrical figure† is the function which, equated to zero, constitutes the equation to such figure. Plücker, I think, somewhere calls it the line or surface function, as the case may be. Geometry, analytically considered, resolves itself into a system of rules for the construction and interpretation of characteristics. One more remark, and I have done. A very comprehensive theorem has been given at the commencement of this commentary, for interpreting the effect of a complete determinant of a linear function of two quadratic functions $(U+\mu V)$, having

* If this be so, then we have the following geometrical theorem :—"*The summit of one of the four cones of the second degree which contain the intersections of two surfaces of the second order drawn in any manner respectively through two given conics lying in the same plane, and having with one another a contact of the third degree, will always be found in the same right line, namely in the tangent line to the two given conics at the point of contact.*"

† More generally, the *characteristic* of any fact or existence is the function which, equated to zero, expresses the condition of the actuality of such fact or existence.

Perhaps the most important pervading principle of modern analysis, but which has never hitherto been articulately expressed, is *that*, according to which we infer, that when one fact of whatever kind is implied in another, the characteristic of the first must contain as a factor the characteristic of the second; and that when two facts are mutually involved, their characteristics will be powers of the same integral function.

The doctrine of characteristics, applied to dependent *systems* of facts, admits of a wide development, logical and analytical.

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one or more pairs of equal factors $(e+\epsilon\mu)$. But here a far wider theory presents itself, of which the aim should be to determine the effect and meaning of this determinant, having any amount and distribution of multiplicity whatsoever among its roots. Nor must our investigations end at that point; but we must be able to determine the meaning and effect of common factors, one or more entering into the successive systems of *minor* determinants derived from the complete determinant of $U+\mu V$.

Nor are we necessarily confined to two, but may take several quadratic functions simultaneously into account.

Aspiring to these wide generalizations, the analysis of quadratic functions soars to a pitch from whence it may look proudly down on the feeble and vain attempts of geometry proper to rise to its level or to emulate it in its flights.

The law which I have stated for assigning the number of independent, or to speak more accurately, non-coevanescent determinants belonging to a given system of minors, I call the Homaloidal law, because it is a corollary to a proposition which represents analytically the indefinite extension of a property common to lines and surfaces to all loci (whether in ordinary or transcendental space) of the first order, all of which loci may, by an abstraction derived from the idea of levelness common to straight lines and planes, be called Homaloids. The property in question is, that neither two straight lines nor two planes can have a common segment; in other words, if n independent relations of rectilinearity or of coplanarity, as the case may be, exist between triadic groups of a series of $n+2$, or between tetradic groups of a series of $n+3$ points respectively, then every triad or tetrad of the series, according to the respective suppositions made, will be in rectilinear or in plane order. So, too, if n independent relations of *coincidence* exist between the duads formed out of $n+1$ points, every duad will constitute a coincidence.

This homaloidal law has not been stated in the above commentary in its form of greatest generality. For this purpose we must commence, not with a square, but with an oblong arrangement of terms consisting, suppose, of m lines and n columns. This will not in itself represent a determinant, but is, as it were, a Matrix out of which we may form various systems of determinants by fixing upon a number p, and selecting at will p lines and p columns, the squares corresponding to which may be termed determinants of the pth order. We have, then, the following proposition. The number of uncoevanescent determinants constituting a system of the pth order derived from a given matrix, n terms broad and m terms deep, may equal, but can never exceed the number

$$(n-p+1)(m-p+1).$$

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Remark on PASCAL'S *and* BRIANCHON'S *Theorems.*

I omitted to state, in the September Number of the *Journal* *, that the demonstration there given by me for Pascal's, applied equally to Brianchon's theorem. This remark is of the more importance, because the fault of the analytical demonstrations hitherto given of these theorems has been, that they make Brianchon's consequence of Pascal's, instead of causing the two to flow simultaneously from the application of the same principles. No demonstration can be held valid in *method*, or as touching the essence of the subject-matter, in which the indifference of the duadic law is departed from. Until these recent times, the analytic method of geometry, as given by Descartes, had been suffered to go on halting as it were on one foot. To Plücker was reserved the honour of setting it firmly on its two equal supports by supplying the complementary system of coordinates. This invention, however, had become inevitable, after the profound views promulgated by Steiner, in the introduction to his Geometry, had once taken hold of the minds of mathematicians. To make the demonstration in the article referred to apply, *totidem literis*, to Brianchon's theorem (recourse being had to the correlative system of coordinates), it is only needful to consider U as the characteristic of the tangential envelope of the conic, x, y, z, t, u, v as the characteristics of the six points of the *circumscribed* hexagon, ϕ the characteristic of the point in which the line x, v meets the line z, t; $ay - \alpha u$ will then be shown to characterize the point in which t, x meets v, z; and thus we see that y, u; t, x; v, z, the three pairs of opposite sides of the hexagon, will meet in one and the same point, which is Brianchon's theorem.

[* p. 138 above.]

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