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On nonlinear stability of incompressible fluid flows

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THE PRESENT paper is concerned with the stability problem of stationary flows of a homogeneous ideal incompressible fluid which can be referred to the classes of motions possessing symmetry of one type (translational, axial, rotational or spiral). The two types of sufficient conditions for nonlinear stability are obtained. The stability conditions for the first type are a generalization for the case of finite amplitude disturbances and new classes of flow of the known Rayleigh criterion concerning the "centrifugal" stability of rotating flows. The second type conditions represent in the same sense a generalization of the other result obtained by Rayleigh. According to this result, the plane parallel fluid flows is stable in the absence of an inflection point of a velocity profile. For systematization the use is made of the analogy between the effects of the density stratification and rotation [1]. The present results concern the stability problem for a wide class of hydrodynamic flows which possesss a necessary symmetry, for example, the flows in stationary or rotating pipes and channels, as well as those with concentrated ring and spiral vortices.

Praca dotyczy problemów stateczności ustalonych przepływów idealnych, nieściśliwych cieczy jednorodnych przy założeniu pewnej symetrii ruchu (translacyjnej, osiowej, obrotowej lub spiralnej). Otrzymano dwa typy warunków dostatecznych nieliniowej stateczności. Warunki stateczności pierwszego typu stanowią uogólnienie znanego kryterium Reyleigha dotyczącego "centryfugalnej" stateczności na przypadek zaburzeń o skończonej amplitudzie i nowych klas przepływów. Warunki drugiego typu są w podobnym sensie uogólnieniem innego wyniku uzyskanego przez Rayleigha. Zgodnie z tym wynikiem płaski przepływ cieczy jest stateczny przy braku punktu przegięcia na profilu prędkości. Zastosowano analogię między efektami stratyfikacji gęstości i obrotu, [1]. Obecne wyniki dotyczą problemu stateczności szerokiej klasy przepływów hydrodynamicznych charakteryzujących się odpowiednią symetrią, jak np. przepływ w ustalonych lub obracających się rurach i kanałach.

Работа касается проблем устойчивости установившихся течений идеальных, несжимаемых однородных жидкостей, при предположении некоторой симметрии движения (трансляционной, осевой, вращательной или спиральной). Получены два типа достаточных условий нелинейной устойчивости. Условия устойчивости первого типа составляют обобщение известного критерия Рэлея, касающегося „центробежной" устойчивости, на случай возмущений конечной амплитуды и новых классов течений. Условия второго типа являются, в аналогичном смысле, обобщением другого результата, полученного Рэлеем. Согласно этому результату плоское течение жидкости является устойчивым при отсутствии точки перегиба на профиле скорости. Применена аналогия между эффектами стратификации плотности и вращения, [1]. Настоящие результаты касаются проблемы устойчивости широкого класса гидродинамических течений, характеризующихся соответствующей симметрией, как например течения в установившихся или вращающихся трубах и каналах.

1. Flows with a spiral symmetry

THE UNSTEADY motions of an ideal incompressible fluid homogeneous in density are considered. In a cylindrical system of coordinates φ, r, z we denote the velocity field components by u, v, w, and p is the pressure field. For spiral symmetry motions u, v, w and p are the functions of three independent variables r, $\mu \equiv a\varphi - bz$ and time t. For example,

$$p = p(r, \mu, t), \tag{1.1}$$

where b is any real number. Without restriction of generality, the parameter a can be considered to take only two values: 0 and 1. When $a = 1$, all solutions of the form (1.1) are 2π-periodic with respect to μ. It is sufficient to consider the values of μ throughout the interval

$$0 \leqslant \mu \leqslant 2\pi. \tag{1.2}$$

When $a = 0$ (the case of a rotational symmetry), the solution can be either periodic or nonperiodic. The notations from [1] are used:

$$\begin{gathered} \alpha \equiv au - brw, \quad \beta \equiv bru + aw, \\ R \equiv a^2 + b^2r^2, \quad g \equiv b^2r/R^2, \quad K \equiv 2ab/R^2. \end{gathered} \tag{1.3}$$

For solutions of the form (1.1) the equations of motion are reduced to

$$\begin{gathered} D(r\alpha/R) + K\beta rv = -p_\mu, \\ Dv - K\beta\alpha - (a\alpha/R)^2/r = -p_r + g\beta^2, \\ D\beta = 0, \quad v + v/r + \alpha_\mu/r = 0, \\ D \equiv \frac{\partial}{\partial t} + v\frac{\partial}{\partial r} + \frac{\alpha}{r}\frac{\partial}{\partial \mu}, \end{gathered} \tag{1.4}$$

where the indices involving the independent variables denote the corresponding partial derivatives.

If the motions (1.1) take place in a fixed region, its boundary must possess a necessary symmetry, i.e. they are prescribed by the function of two variables:

$$f(r, \mu) = 0. \tag{1.5}$$

The nonpenetration conditions for u, v, w on the boundary prescribed by the relation (1.5) which were written down in Eqs. (1.3), and (1.5) give

$$vf_r + \frac{\alpha}{r}f_\mu = 0. \tag{1.6}$$

The important particular case of Eq. (1.6) is circular cylindrical boundaries with the flow region

$$R_1 < r < R_2. \tag{1.7}$$

When $a = 1$, it is convenient to consider the boundary conditions (1.6) and Eq. (1.4) on the plane of polar coordinates r, μ, where r is the radial coordinate and μ is the angular one (see the relation (1.2)). On this plane the closed curves (1.5) bound the flow region τ. Under such conditions the expressions (1.4)–(1.6) turned out to be rather similar to the equations and boundary conditions for the plane motions of the density-inhomogeneous (stratified) fluid [1]. The value of α takes the role of the μ-component of velocity, and the variable β plays the role of density. The corresponding "field of mass forces" is directed along the radius from the center and has a value of g (1.3). For the rotational-symmetric motions ($a = 0$) this similarity transforms to equivalence [1].

For the general case of the problem (1.4)–(1.6) the similarity of the relation (1.1) with the motions of an inhomogeneous fluid is so considerable that the integral of energy

E has the form of a sum of the fictitious "kinetic" energy T and "potential" one Π: $E = T+\Pi$

(1.8)
$$T \equiv \int_\tau \left(\frac{\alpha^2}{R} + v^2\right) d\tau, \qquad d\tau \equiv r\,dr\,d\mu,$$
$$\Pi \equiv \int_\tau \beta^2 U d\tau, \qquad U = U(r) \equiv \int_0^r g(\xi)d\xi.$$

In the terms of the velocity components u, v, w the value of E represents the kinetic energy taken for one period of the relation (1.2). The other integral of Eqs. (1.4)–(1.6) is determined by the arbitrary functions $\Phi(\beta)$:

(1.9)
$$I \equiv \int_\tau \Phi(\beta) d\tau = \text{const}.$$

2. Analogies of states of hydrostatic equilibrium

The problem (1.4)–(1.6) has the exact solutions (with an arbitrary function $\beta_0(r)$) for the "state of rest" in the form

(2.1)
$$\alpha = v = 0, \qquad \beta = \beta_0(r).$$

In terms of the velocity components u, v, w the solution (2.1) corresponds to the flow

(2.2)
$$u = u_0(r), \qquad v = 0, \qquad w = w_0(r); \qquad au_0 = brw_0$$

which is prescribed by an arbitrary function $u_0(r)$.

For the rotational symmetric motions ($a = 0$) the equivalents of the hydrostatic states are the circular flows

(2.3)
$$u = u_0(r), \qquad v = 0, \qquad w = 0,$$

where the function $u_0(r)$ within the interval (1.7) is arbitrary.

Now let

(2.4)
$$\alpha = \alpha(r, \mu, t), \qquad v = v(r, \mu, t), \qquad \beta = \beta(r, \mu, t)$$

be an exact nonstationary solution of the problem (1.4)–(1.6) considered as a disturbance of the state of rest (2.1). The following theorem is valid.

THEOREM 1. *Let throughout the whole region τ of the flow* (2.1) *the inequality with constants c^- and c^+ be satisfied*:

(2.5)
$$0 < c^- \leqslant g/(\beta_0^2)_r \leqslant c^+ < \infty.$$

Then the disturbances α, v, $\sigma \equiv \beta^2 - \beta_0^2$ of the flow (2.1) *are estimated by their initial values α_*, v_*, σ_* as follows*:

(2.6)
$$\int_\tau \left(\frac{\alpha^2}{R} + v^2 + c^-\sigma^2\right) d\tau \leqslant \int_\tau \left(\frac{\alpha_*^2}{R} + v_*^2 + c^+\sigma_*^2\right) d\tau.$$

Proof. The notations $\varrho \equiv \beta^2$, $\varrho_0 \equiv \beta_0^2$ are adopted. From Eqs. (1.9), and (1.10) a conserving functional has the form

$$F(\alpha, v, \varrho) = \int\limits_{\tau} \left[\frac{\alpha^2}{R} + v^2 + \varrho U + \Phi(\varrho)\right] d\tau$$

which is presented in the form of three summands:

$$F(\alpha, v, \varrho) \equiv F(0, 0, \varrho_0) + F_1 + F_2,$$

$$F_1 \equiv \int\limits_{\tau} \sigma[\varphi(\varrho_0) + \Phi'(\varrho_0)]\, d\tau,$$

$$F_2 \equiv \int\limits_{\tau} \left[\frac{1}{2}\left(\frac{\alpha^2}{R} + v^2\right) + \Phi(\varrho_0 + \sigma) - \Phi(\varrho_0) - \Phi'(\varrho_0)\sigma\right] d\tau.$$

The upper prime denotes an ordinary derivative. In F_1 the function $U(r)$ is substituted by $U = \varphi(\varrho_0)$ which is obtained when r is excluded from $U = U(r)$ Eq. (1.9) and $\varrho = \varrho_0(r)$ the flow (2.1). By virtue of the inequality (2.5) the function $\varphi(\varrho_0)$ is monotone. Using an arbitrariness of the function $\Phi(\varrho)$, we can assume $\Phi'(\varrho_0) \equiv -\varphi(\varrho_0)$. As a result, $F_1 \equiv 0$, and the functional F_2 is time-independent.

Since

$$\varphi'(\varrho_0) = \frac{dU}{dr} \Big/ \frac{d\varrho_0}{dr} = -g/(\beta_0^2)_r$$

then the inequality (2.5) gives the inequality

$$c^- \leqslant \Phi'' \leqslant c^+ \tag{2.7}$$

which is satisfied within an interval of ϱ_0 change in the region τ. Let the function $\Phi(\varrho)$ be determined additionally for all remaining values of ϱ, so that the property (2.7) be valid. Then for any of the two values of h and l by integrating the inequality (2.7), it is obtained that

$$\frac{1}{2} c^- l^2 \leqslant \Phi(h+l) - \Phi(h) - \Phi'(h) l \leqslant \frac{1}{2} c^+ l^2. \tag{2.8}$$

Now from the inequality (2.8) and the conservation value of F_2 we derive the relation (2.6). At rotational symmetry of the motions ($a = 0$), the estimate (2.6) for the disturbance of the flow (2.3), (1.8) is reduced to

$$\int\limits_{R_1}^{R_2} (v^2 + w^2 + c^- \sigma^2) r\, dr \leqslant \int\limits_{R_1}^{R_2} (v_*^2 + w_*^2 + c^+ \sigma_*^2) r\, dr, \tag{2.9}$$

where $\sigma \equiv r^2(u^2 - u_0^2)$; the condition (2.5) gives

$$c^- \leqslant [r^3 (r^2 u_0^2)_r]^{-1} \leqslant c^+. \tag{2.10}$$

The upper bounds of the arbitrary disturbance by the initial data (2.6), (2.9) indicate the root-mean-square stability of the solution (2.1)–(2.3) in the sense of the Lyapunov definition.

The bound (2.9), (2.10) represents a nonlinear version of the Rayleigh criterion which is well-known in the linear stability theory. It guarantees the stability of the rotational symmetric flow (2.3) with respect to the same type disturbances if the circulation square $r^2u_0^2$ increases with radius r. Theorem 1 gives a nonlinear analogy of the Rayleigh criterion for the flows (2.2) with a more complicated (spiral) geometry.

3. Analogies of plane motions of a homogeneous fluid

By virtue of the equation $D\beta = 0$, the solutions of the system (1.4) with $\beta = \text{const}$ form an independent class. If the initial data are prescribed in this class, the solution also belongs to it. In this case the time derivative order of the system of equations (1.4) decreases by unit, it becomes the first-order system. After elimination of pressure from the first two equations (1.4), the following system is obtained:

$$D\lambda = 0, \quad v_r + v/r + \alpha_\mu/r = 0,$$

(3.1)
$$\lambda \equiv \omega + \frac{2ab\beta}{R}, \quad \omega \equiv \frac{1}{r}\left[\left(\frac{r\alpha}{R}\right)_r - v_\mu\right].$$

Then, after introducing the stream function $\psi(rv = -\psi_\mu, \alpha = \psi_r)$ the relation (3.1) is reduced to one equation of ψ. The boundary condition (1.6) takes the form of $\psi = \text{const}$ in Eq. (1.5).

For the class of motions under consideration the integral I (1.9) is trivial and useless. At the same time there is the other integral determined through the arbitrary functions $\Phi(\lambda)$:

(3.2)
$$I = \int_\tau \Phi(\lambda) d\tau = \text{const}.$$

Let the certain stationary solution of Eqs. (3.1) and (1.6) be as follows:

(3.3)
$$\psi = \psi_0(r, \mu), \quad \beta = \beta_0 = \text{const}; \quad \lambda = \lambda_0(r, \mu, \beta_0).$$

The equation $D\lambda = 0$ gives the functional dependence $\psi_0 = \Psi(\lambda_0)$. Then, let

$$\psi(r, \mu, t) = \psi_0 + \varphi(r, \mu, t), \quad \lambda(r, \mu, t) = \lambda_0 + \varkappa(r, \mu, t)$$

be the certain nonstationary solution of Eq. (3.1) and (1.6) considered as a disturbance of the flow (3.3).

THEOREM 2. *Let throughout the region τ of the flow* (3.3) *the inequality with constants c^- and c^+ be satisfied*:

(3.4)
$$0 < c^- \leqslant d\Psi/d\lambda_0 \leqslant c^+ < \infty.$$

Then the disturbances φ, $\varkappa$ are estimated by their initial values φ^*, $\varkappa^*$ as follows:

(3.5)
$$\int_\tau \left(\frac{\varphi_r^2}{R} + \frac{\varphi_\mu^2}{r^2} + c^-\varkappa^2\right) d\tau \leqslant \int_\tau \left(\frac{\varphi_{*r}^2}{R} + \frac{\varphi_{*\mu}^2}{r^2} + c^+\varkappa_*^2\right) d\tau.$$

The proof is based on consideration of the integrals (1.8), (3.2) and (3.3) and is not given here. The result obtained can be essentially improved for the important cases of the rotational symmetric solutions (3.3) and circular geometry of the boundaries (1.7). In terms of the velocity components u, v, w, such flows are prescribed by two interconnected functions of:

(3.6) $$u = u_0(r), \quad v = 0, \quad w = w_0(r); \quad bru_0 + aw_0 = \beta_0.$$

Such flows and the nonpenetration conditions on the boundary (1.7) are invariant with respect to the translation along the axis z. This allows us to consider the stability problem in any coordinate system moving along the z-axis with constant velocity M. As a result, the same Theorem 2 is obtained in which the inequality (3.4) takes a more useful and concrete form

(3.7) $$c^- \leqslant \frac{d\Psi}{d\lambda_0} = \frac{\alpha_0 + brM}{A} \leqslant c^+.$$

Here $\alpha_0 \equiv au_0 - brw_0$. The value $A \equiv d\lambda_0/dr$ is independent of M and is prescribed by

(3.8) $$A = \left[\frac{(ru_0)_r}{ar}\right]_r = -\left(\frac{w_0 r}{br}\right)_r.$$

THEOREM 3. *If there exists such constant M that throughout the interval* (1.7) *the inequality* (3.7) *is fulfilled, then the flow* (3.6) *is stable in the root-mean-square* (3.5).

This stability condition comprises, in particular, the following Theorem.

THEOREM 4. *If the continuous function* $A(r)$ (3.8) *has no zero values within the interval* (1.7), *then the flow* (3.6) *is stable.*

Theorems 2–4 are a generalization for new classes of motions (1.1) and finite-amplitude disturbances of the well-known Rayleigh result concerning the stability of a parallel flow in the absence of the inflection point in the velocity profile. In the particular case ($b = 0$), the motions of the class (1.1) are plane and Theorems 2–4 give the results obtained previously by Rayleigh, Fjortoft and Arnold. When $a = 0$ (for the problem of stability of the axisymmetric flow in a circular pipe), the linear version of Theorem 4 is also obtained by Rayleigh.

4. Rotating flows with translational symmetry

The motions of a homogeneous fluid are considered in the coordinate system rotating with the constant velocity $\mathbf{\Omega}/2$. The equations of motion are written as follows:

(4.1) $$D\mathbf{u} + \mathbf{\Omega} \times \mathbf{u} = -\nabla p^*,$$
$$\operatorname{div}\mathbf{u} = 0, \quad D \equiv \frac{\partial}{\partial t} + \mathbf{u}\cdot\nabla,$$

where $\mathbf{u}$ is the velocity vector, p^* is the modified pressure inclusing the "centrifugal" addition. Let $\mathbf{k}$ be a unique vector which prescribes the fixed (in the rotating system) direction and forms an angle θ ($0 \leqslant \theta \leqslant \pi$) with the vector $\mathbf{\Omega}$. The class of solutions of Eqs. (4.1) where $\mathbf{u}$ and p^* are invariable in the $\mathbf{k}$-direction is investigated. The system of the Cartesian coordinates x, y, z is introduced so that the z-axis is parallel to the $\mathbf{k}$-vector and the vector

$\boldsymbol{\Omega}$ lies in the plane of x, z. For the considered motions the field of velocities $\mathbf{u} = (u, v, w)$ and pressure p^* are independent of the coordinate z:

$$\mathbf{u} = \mathbf{u}(x, y, t), \quad p^* = p^*(x, y, t). \tag{4.2}$$

If we introduce the notations [1]

$$\begin{aligned} \Omega &= (\Omega_1, 0, \Omega_3), \quad \varrho \equiv w - \Omega_1 y, \\ g &= k \times \Omega = (0, g, 0), \quad g \equiv \Omega_1, \end{aligned} \tag{4.3}$$

the system of equations (4.1) for the motions (4.2) may be reduced to

$$\begin{gathered} Du = -p_x, \quad Dv = -p_y + \varrho g, \\ D\varrho = 0, \quad u_x + v_y = 0, \quad D \equiv \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}, \end{gathered} \tag{4.4}$$

where $p \equiv p^* - \Omega_3 \psi + \Omega_1^2 y^2/2$; ψ is the stream function for which $u = -\psi_y$, $v = \psi_x$. If the motion (4.2) takes place in the fixed region, its boundary must have the form of a cylindrical surface with a generatrix parallel to the z-axis, i.e. it is prescribed by

$$f(x, y) = 0. \tag{4.5}$$

On the plane x, y the curve (4.5) restricts the flow region τ. The nonpenetration conditions on the boundary (4.5) give

$$uf_x + vf_y = 0. \tag{4.6}$$

It is remarkable that the expressions (4.4)–(4.6) coincide with the equations and the corresponding boundary conditions in the case of the plane motions of an inhomogeneous (stratified) fluid in the Boussinesq approximation. The integral of kinetic energy E for Eqs. (4.1) and (4.6) in terms of (4.3), (4.4) is written in the form of a sum of the fictitious "kinetic" and "potential" energies T and Π:

$$\begin{gathered} E = T + \Pi = \text{const}, \\ T \equiv \frac{1}{2}\int_\tau (u^2 + v^2)\, d\tau, \quad \Pi \equiv \int_\tau \varrho U\, d\tau, \quad d\tau \equiv dx\, dy, \end{gathered} \tag{4.7}$$

where U is the potential introduced according to $\mathbf{g} = -\nabla U$. The other integral of Eqs. (4.4)–(4.6) is

$$I = \int_\tau \Phi(\varrho)\, d\tau \tag{4.8}$$

with the arbitrary function $\Phi(\varrho)$.

The analogies of states of the hydrostatic equilibrium in the class (4.2) are the exact solutions of Eqs. (4.4) having the form

$$u = v = 0, \quad \varrho = \varrho_0(y). \tag{4.9}$$

In the initial terms of Eqs. (4.1) the parallel flow is prescribed by the relations (4.9):

$$u = v = 0, \quad w = w_0(y). \tag{4.10}$$

The functions $\varrho_0(y)$ and $w_0(y)$ in Eqs. (4.9) and (4.10) are arbitrary. Now let

$$u = u(x, y, t), \quad v = v(x, y, t), \quad \varrho = \varrho_0(y) + \sigma(x, y, t)$$

be an exact nonstationary solution of Eqs. (4.4)–(4.6) considered as a disturbance of "the state of rest" (4.9). The next Theorem is valid:

THEOREM 5. *Let throughout the region* τ *the inequality with constants* c^- *and* c^+ *be satisfied*:

$$0 < c^- \leqslant g/\varrho_{0y} \leqslant c^+ < \infty .$$

Then the disturbances u, v, σ *of the flow* (4.9), (4.10) *are estimated by their initial values as follows*

$$\int_\tau (u^2 + v^2 + c^- \sigma^2) d\tau \leqslant \int_\tau (u_*^2 + v_*^2 + c^+ \sigma_*^2) d\tau .$$

The proof is based on the availability of the integrals E (4.7) and I (4.8). Theorem 5 is the analogy of the Rayleigh criterion of centrifugal stability for the translational invariant motions. In conclusion it should be noted that all the above-mentioned statements concerning stability are conditional in the sense that the stability is guaranteed only for the special classes of disturbances which possess the same symmetry as main flows. The stability proofs in such classes have evidently a limited physical significance. However, the difficulties of investigating the nonlinear hydrodynamic problems are so significant that the information about the properties of particular classes of motions is, to the author's opinion, of indubitable interest.

REMARK. The proofs of Theorems (1, 2, 5) are performed by the method presented in [2].

References

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