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ON THE RELATION BETWEEN THE MINOR DETERMINANTS OF LINEARLY EQUIVALENT QUADRATIC FUNCTIONS.

[*Philosophical Magazine*, I. (1851), pp. 295—305.]

I SHOWED in the preliminary part of my paper on Contacts in the February Number of this *Magazine**, by *à priori* reasoning, that if a quadratic function (U) be linearly converted into another (V), any minor determinant of any order of V must be a syzygetic function of all the minor determinants of U of the same order.

The object of my present communication is to exhibit the syzygy in question, which, as I indicated, is linear; by which I mean that a determinant of the one function is equal to the sum of the pari-ordinal determinants of the other affected respectively with multipliers formed exclusively out of the coefficients of the equations of transformation. In order that a clear enunciation of the theorem in view may be possible, it is necessary to premise a new but simple, and, as experience has proved to me, a most powerful, because natural, method of notation applicable to all questions concerning determinants.

Every determinant is obtained by operating upon a square array of quantities, which, according to the ordinary method, might be denoted as follows:

$$\begin{array}{l} a_{1,1},\quad a_{1,2}\ldots a_{1,n},\\ a_{2,1},\quad a_{2,2}\ldots a_{2,n},\\ a_{3,1},\quad a_{3,2}\ldots a_{3,n},\\ \ldots\ldots\ldots\ldots\ldots\ldots\\ a_{n,1},\quad a_{n,2}\ldots a_{n,n}. \end{array}$$

My method consists in expressing the same quantities biliterally as below:

$$\begin{array}{l} a_1\alpha_1,\quad a_1\alpha_2\ldots a_1\alpha_n,\\ a_2\alpha_1,\quad a_2\alpha_2\ldots a_2\alpha_n,\\ \ldots\ldots\ldots\ldots\ldots\ldots\\ a_n\alpha_1,\quad a_n\alpha_2\ldots a_n\alpha_n, \end{array}$$

[* p. 221 above.]

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where of course, whenever desirable, instead of $a_1, a_2 \dots a_n$, and $\alpha_1, \alpha_2 \dots \alpha_n$, we may write simply $a, b \dots l$, and $\alpha, \beta \dots \lambda$ respectively. Each quantity is now represented by two letters; the letters themselves, taken separately, being symbols neither of quantity nor of operation, but mere umbræ or ideal *elements* of quantitative symbols. We have now a means of representing the determinant above given in a compact form; for this purpose we need but to write one set of umbræ over the other as follows: $\begin{pmatrix} a_1, a_2 \dots a_n \\ \alpha_1, \alpha_2 \dots \alpha_n \end{pmatrix}$. If we now wish to obtain the algebraic value of this determinant, it is only necessary to take $\alpha_1, \alpha_2 \dots \alpha_n$ in all its 1, 2, 3 ... n different positions, and we shall have

$$\begin{Bmatrix} a_1, & a_2 \dots a_n \\ \alpha_1, & \alpha_2 \dots \alpha_n \end{Bmatrix} = \Sigma \pm \{a_1\alpha_{\theta_1} \times a_2\alpha_{\theta_2} \times \dots \times a_n\alpha_{\theta_n}\},$$

in which expression $\theta_1, \theta_2 \dots \theta_n$ represents some order of the numbers $1, 2 \dots n$, and the positive or negative sign is to be taken according to the well-known dichotomous law. Thus, for example,

$$\begin{Bmatrix} a\,b\,c \\ \alpha\beta\gamma \end{Bmatrix} \text{ will represent } \left.\begin{array}{r} a\alpha \times b\beta \times c\gamma \\ +\, a\beta \times b\gamma \times c\alpha \\ +\, a\gamma \times b\alpha \times c\beta \\ -\, a\beta \times b\alpha \times c\gamma \\ -\, a\alpha \times b\gamma \times c\beta \\ -\, a\gamma \times b\beta \times c\alpha \end{array}\right\}.$$

Although not necessary for our immediate object, it may not be inopportune to observe how readily this notation lends itself to a further natural extension of its application.

$\begin{Bmatrix} \overline{ab} & \overline{cd} \\ \alpha\beta & \gamma\delta \end{Bmatrix}$ will naturally denote

$$\frac{ab}{\alpha\beta} \times \frac{cd}{\gamma\delta} - \frac{ab}{\gamma\delta} \times \frac{cd}{\alpha\beta};$$

that is

$$\begin{Bmatrix} (a\alpha \times b\beta) \\ -\,(a\beta \times b\alpha) \end{Bmatrix} \times \begin{Bmatrix} (c\gamma \times d\delta) \\ -\,(c\delta \times d\gamma) \end{Bmatrix} - \begin{Bmatrix} (a\gamma \times b\delta) \\ -\,(a\delta \times b\gamma) \end{Bmatrix} \times \begin{Bmatrix} (c\alpha \times d\beta) \\ -\,(c\beta \times d\alpha) \end{Bmatrix}.$$

And in general the compound determinant

$$\begin{Bmatrix} \overline{a_1, b_1 \dots l_1}, & \overline{a_2, b_2 \dots l_2} \dots & \overline{a_r, b_r \dots l_r} \\ \alpha_1, \beta_1 \dots \lambda_1, & \alpha_2, \beta_2 \dots \lambda_2 & \alpha_r, \beta_r \dots \lambda_r \end{Bmatrix}$$

will denote

$$\Sigma \pm \begin{Bmatrix} a_1, & b_1 \dots l_1 \\ \alpha_{\theta_1}, & \beta_{\theta_1} \dots \lambda_{\theta_1} \end{Bmatrix} \times \begin{Bmatrix} a_2, & b_2 \dots l_2 \\ \alpha_{\theta_2}, & \beta_{\theta_2} \dots \lambda_{\theta_2} \end{Bmatrix} \times \dots \times \begin{Bmatrix} a_r, & b_r \dots l_r \\ \alpha_{\theta_r}, & \beta_{\theta_r} \dots \lambda_{\theta_r} \end{Bmatrix},$$

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where, as before, we have the disjunctive equation

$$\theta_1,\ \theta_2 \ldots \theta_r = 1,\ 2 \ldots r.$$

As an example of the power of this notation, I will content myself with stating the following remarkable theorem in compound determinants, one of the most prolific in results of any with which I am acquainted, but which is derived from a more particular case of another vastly more general. The theorem is contained in the annexed equation

$$\left\{\begin{matrix} \overline{a_1,\ a_2 \ldots a_r,\ a_{r+1},} & \overline{a_1,\ a_2 \ldots a_r,\ a_{r+2}} \cdots & \overline{a_1,\ a_2 \ldots a_r,\ a_{r+s}} \\ \alpha_1,\ \alpha_2 \ldots \alpha_r,\ \alpha_{r+1}, & \alpha_1,\ \alpha_2 \ldots \alpha_r,\ \alpha_{r+2} & \alpha_1,\ \alpha_2 \ldots \alpha_r,\ \alpha_{r+s} \end{matrix}\right\}$$

$$= \begin{Bmatrix} a_1,\ a_2 \ldots a_r \\ \alpha_1,\ \alpha_2 \ldots \alpha_r \end{Bmatrix}^{s-1} \times \begin{Bmatrix} a_1,\ a_2 \ldots a_r,\ a_{r+1},\ a_{r+2} \ldots a_{r+s} \\ \alpha_1,\ \alpha_2 \ldots \alpha_r,\ \alpha_{r+1},\ \alpha_{r+2} \ldots \alpha_{r+s} \end{Bmatrix}. \quad (1)$$

It is obvious, that, without the aid of my system of umbral or biliteral notation, this important theorem could not be made the subject of statement without an enormous periphrasis, and could never have been made the object of distinct contemplation or proof.

To return to the more immediate object of this communication, suppose that we have any binary function of two sets of quantities, $x_1,\ x_2 \ldots x_n$; $\xi_1,\ \xi_2 \ldots \xi_n$, of which the general term will be of the form $c_{r,s} \times x_r\,\xi_s$; according to the principles of notation above laid down, nothing can be more natural than to represent $c_{r,s}$ by the biliteral group $a_r\alpha_s$; the function in question will then take the form

$$\Sigma a_r\alpha_s \,.\, x_r\xi_s\,;$$

the x's and ξ's denoting quantities, but the a's and α's mere umbræ. The function may then be thrown under the convenient symbolical form

$$(a_1x_1 + a_2x_2 + \ldots + a_n x_n)$$
$$\times\,(\alpha_1\xi_1 + \alpha_2\xi_2 + \ldots + \alpha_n\,\xi_n).$$

So if we confine ourselves to quadratic functions, for which $x_1,\ x_2 \ldots x_n$; $\xi_1,\ \xi_2 \ldots \xi_n$ become respectively identical, the general symbolical representation of any such will be

$$(a_1x_1 + a_2x_2 + \ldots + a_nx_n)^2.$$

The complete determinant will be denoted by

$$\begin{Bmatrix} a_1,\ a_2 \ldots a_n \\ \alpha_1,\ \alpha_2 \ldots \alpha_n \end{Bmatrix},$$

and any minor determinant of the rth order by

$$\begin{Bmatrix} a_1,\ a_2\ \ldots\ a_r \\ \alpha_{\theta_1},\ \alpha_{\theta_2}\ \ldots\ \alpha_{\theta_r} \end{Bmatrix},$$

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where $\theta_1, \theta_2 \ldots \theta_r$ are some certain r distinct numbers taken out of the series $1, 2, 3 \ldots r$. Suppose now that we have

$$U = (a_1x_1 + a_2x_2 + \ldots + a_nx_n)^2$$

linearly transformable into

$$V = (b_1y_1 + b_2y_2 + \ldots + b_ny_n)^2,$$

by means of the n equations

$$\left.\begin{array}{l} x_1 = a_1b_1 . y_1 + a_1b_2 . y_2 + \ldots + a_1b_n . y_n \\ x_2 = a_2b_1 . y_1 + a_2b_2 . y_2 + \ldots + a_2b_n . y_n \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ x_n = a_nb_1 . y_1 + a_nb_2 . y_2 + \ldots + a_nb_n . y_n \end{array}\right\};$$

in which equations, be it observed, each coefficient a_rb_s is a single quantity, perfectly independent of the quantities denoted generally by a_ra_s, b_rb_s which enter into U and V. Our object is to be able to express the minor determinant

$$\begin{Bmatrix} b_{k_1}, & b_{k_2} \ldots b_{k_r} \\ b_{l_1}, & b_{l_2} \ldots b_{l_r} \end{Bmatrix},$$

in which the one group of distinct numbers, $k_1, k_2 \ldots k_r$ may either differ wholly from, or agree wholly or in part with the other group of distinct numbers $l_1, l_2 \ldots l_r$, under the form of

$$\Sigma \left\{ \begin{pmatrix} a_{\theta_1}, & a_{\theta_2} \ldots a_{\theta_r} \\ b_{\phi_1}, & b_{\phi_2} \ldots b_{\phi_r} \end{pmatrix} \times Q \right\}.$$

The particular value of Q corresponding to each double group, $\begin{pmatrix} \theta_1, & \theta_2 \ldots \theta_r \\ \phi_1, & \phi_2 \ldots \phi_r \end{pmatrix}$, may be denoted by $Q\begin{pmatrix} \theta_1, & \theta_2 \ldots \theta_r \\ \phi_1, & \phi_2 \ldots \phi_r \end{pmatrix}$; so that our problem consists in determining the value of $Q\begin{pmatrix} \theta_1, & \theta_2 \ldots \theta_r \\ \phi_1, & \phi_2 \ldots \phi_r \end{pmatrix}$ in the equation

$$\begin{Bmatrix} b_{k_1}, & b_{k_2} \ldots b_{k_r} \\ b_{l_1}, & b_{l_2} \ldots b_{l_r} \end{Bmatrix} = \Sigma \left\{ Q\begin{pmatrix} \theta_1, & \theta_2 \ldots \theta_r \\ \phi_1, & \phi_2 \ldots \phi_r \end{pmatrix} \times \begin{pmatrix} a_{\theta_1}, & a_{\theta_2} \ldots a_{\theta_r} \\ a_{\phi_1}, & a_{\phi_2} \ldots a_{\phi_r} \end{pmatrix} \right\}.$$

Accordingly I enunciate that

$$Q\begin{pmatrix} \theta_1, & \theta_2 \ldots \theta_r \\ \phi_1, & \phi_2 \ldots \phi_r \end{pmatrix} = \begin{Bmatrix} a_{k_1}, & a_{k_2} \ldots a_{k_r} \\ b_{\theta_1}, & b_{\theta_2} \ldots b_{\theta_r} \end{Bmatrix} \times \begin{Bmatrix} a_{l_1}, & a_{l_2} \ldots a_{l_r} \\ b_{\phi_1}, & b_{\phi_2} \ldots b_{\phi_r} \end{Bmatrix}$$

$$+ \begin{Bmatrix} a_{l_1}, & a_{l_2} \ldots a_{l_r} \\ b_{\theta_1}, & b_{\theta_2} \ldots b_{\theta_r} \end{Bmatrix} \times \begin{Bmatrix} a_{k_1}, & a_{k_2} \ldots a_{k_r} \\ b_{\phi_1}, & b_{\phi_2} \ldots b_{\phi_r} \end{Bmatrix}, \qquad (2)$$

subject to one sole exception in the case of $\theta_1, \theta_2 \ldots \theta_r$ being identical with $\phi_1, \phi_2, \ldots \phi_r$; namely, that for the terms (for such case) of the form

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$Q\begin{pmatrix}\theta_1, & \theta_2 \dots \theta_r \\ \theta_1, & \theta_2 \dots \theta_r\end{pmatrix}$, the value to be taken is not that which the general formula would give, namely,

$$2\begin{Bmatrix}a_{k_1}, & a_{k_2} \dots a_{k_r} \\ b_{\theta_1}, & b_{\theta_2} \dots b_{\theta_r}\end{Bmatrix}^2,$$

but the half of this, that is simply the square of

$$\begin{Bmatrix}a_{k_1}, & a_{k_2} \dots a_{k_r} \\ a_{\theta_1}, & a_{\theta_2} \dots a_{\theta_r}\end{Bmatrix}.$$

The value of $Q\begin{pmatrix}\theta_1, & \theta_2 \dots \theta_r \\ \phi_1, & \phi_2 \dots \phi_r\end{pmatrix}$, it is obvious, contains only *quantities* of the form $a_r \,.\, b_s$, which are coefficients in the equations of transformation, but none of the form $a_r \,.\, a_s$ or $b_r \,.\, b_s$; showing that the syzygetic connexion between the minor determinants of U and V of the same order is linear, as has been already anticipatively announced.

The problem which I have treated above is only a particular case of a more general one, which may be stated as follows: given

$$U = (a_1x_1 + a_2x_2 + \dots + a_nx_n)^2,$$

and supposing m linear equations to be instituted between $x_1, x_2 \dots x_n$, so that U may be made a function of $(n-m)$ letters only, to express any minor determinant of the reduced form of U without performing the process of elimination between the given equations. Let the given equations be written under the form

$$a_1a_{n+1}x_1 + a_2a_{n+1}x_2 + \dots + a_na_{n+1}x_n = 0,$$
$$a_1a_{n+2}x_1 + a_2a_{n+2}x_2 + \dots + a_na_{n+2}x_n = 0,$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$
$$a_1a_{n+m}x_1 + a_2a_{n+m}x_2 + \dots + a_na_{n+m}x_n = 0,$$

and let it be convened (which takes nothing away from the generality of these equations) that $a_{n+r}a_{n+s}$ shall signify zero for all values of r and s *concurrently* greater than zero. Suppose that $x_1, x_2 \dots x_m$, being eliminated, U becomes of the form

$$(b_{m+1}x_{m+1} + b_{m+2}x_{m+2} + \dots + b_nx_n)^2;$$

and suppose that we wish to determine the value of the complete determinant of this last function; it will be found to be

$$\begin{pmatrix}b_{m+1}, & b_{m+2} \dots b_n \\ b_{m+1}, & b_{m+2} \dots b_n\end{pmatrix} = \begin{Bmatrix}a_1, & a_2 \dots a_n, & a_{n+1} \dots a_{n+m} \\ a_1, & a_2 \dots a_n, & a_{n+1} \dots a_{n+m}\end{Bmatrix} \div \begin{pmatrix}a_1, & a_2 & \dots a_m \\ a_{n+1}, & a_{n+2} & \dots a_{n+m}\end{pmatrix}^2,$$

the squared divisor being, as is obvious, a function only of the coefficients of the transforming equations, and depending for its value upon the particular

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m quantities selected for elimination. The dividend, on the contrary, is independent of this selection, but involves the coefficients of the function combined with the coefficients of transformation. This is the symbolical representation of the theorem given by me in the postscript to my paper in the *Cambridge and Dublin Mathematical Journal* for November 1850*.

Suppose, now, more generally that we wish to find any minor determinant. The solution is given† by the equation

$$\begin{Bmatrix} b_{\theta_{m+1}}, & b_{\theta_{m+2}} & \cdots & b_{\theta_{m+s}} \\ b_{\phi_{m+1}}, & b_{\phi_{m+2}} & \cdots & b_{\phi_{m+s}} \end{Bmatrix}$$

(wherein the two groups $\theta_{m+1}, \theta_{m+2}, \ldots \theta_{m+s}$; $\phi_{m+1}, \phi_{m+2} \cdots \phi_{m+s}$ are each of them s differing, or wholly or in part agreeing individuals arbitrarily selected out of the $(n-m)$ numbers $m+1$, $m+2$, ... n)

$$= \begin{Bmatrix} a_{\theta_1}, & a_{\theta_2} & \cdots & a_{\theta_m}, & a_{\theta_{m+1}}, & a_{\theta_{m+2}} & \cdots & a_{\theta_{m+s}} \\ a_{\phi_1}, & a_{\phi_2} & \cdots & a_{\phi_m}, & a_{\phi_{m+1}}, & a_{\phi_{m+2}} & \cdots & a_{\phi_{m+s}} \end{Bmatrix} \div \begin{Bmatrix} a_1, & a_2 & \cdots a_m \\ a_{n+1}, & a_{n+2} & \cdots a_{n+m} \end{Bmatrix}^2. \qquad (3)$$

If we make $n = 2\gamma$ and $m = \gamma$, and $a_{\gamma+r}a_{\gamma+s} = 0$ for all positive values of either r or s, and $a_{\gamma-i}a_{n+e} = 0$ for all values of i and e differing from one another, and for equal values $a_{\gamma-e}a_{\gamma+c} = -1$, it will readily be seen that this last theorem reduces to the one first considered; and on careful inspection it will be found, that the solution given of the general question includes within it that presented for the particular case in question. Such inclusion, however, I ought in fairness to state is far from being obvious; and to demonstrate it exactly, and in general terms, requires the aid of methods which my readers would probably find to exceed their existing degree of knowledge or familiarity with the subject.

The theorem above enunciated was in part suggested in the course of a conversation with Mr Cayley (to whom I am indebted for my restoration to the enjoyment of mathematical life) on the subject of one of the preliminary theorems in my paper on Contacts in this *Magazine*.

It is wonderful that a theory so purely analytical should originate in a geometrical speculation. My friend M. Hermite has pointed out to me, that some faint indications of the same theory may be found in the *Recherches Arithmétiques* of Gauss. The notation which I have employed for determinants is very similar to that of Vandermonde, with which I have become acquainted since writing the above, in Mr Spottiswoode's valuable treatise *On the Elementary Theorems of Determinants*. Vandermonde was evidently on the right road. I do not hesitate to affirm, that the superiority of his and my notation over that in use in the ordinary methods is as great and almost as important to the progress of analysis, as the superiority of the notation of the differential calculus over that of the fluxional system. For what is the theory of determinants? It is an algebra upon algebra; a

[* p. 136 above.] [† see p. 251 below.]

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calculus which enables us to combine and foretell the results of algebraical operations, in the same way as algebra itself enables us to dispense with the performance of the special operations of arithmetic. All analysis must ultimately clothe itself under this form*.

I have in previous papers defined a "Matrix" as a rectangular array of terms, out of which different systems of determinants may be engendered, as from the womb of a common parent; these cognate determinants being by no means isolated in their relations to one another, but subject to certain simple laws of mutual dependence and simultaneous deperition. The condensed representation of any such Matrix, according to my improved Vandermondian notation, will be

$$\begin{Bmatrix} a_1, & a_2 \ldots a_n \\ \alpha_1, & \alpha_2 \ldots \alpha_m \end{Bmatrix}.$$

To return to the theorems of the text. Theorem (2) admits of being presented in a more convenient form for the purposes of analytical operation, so as to become relieved from all cases of exception appertaining to particular terms.

The limitation to the generality of the expression for Q arises from our treating

$$\begin{Bmatrix} a_{\theta_1}, & a_{\theta_2} \ldots a_{\theta_r} \\ a_{\phi_1}, & a_{\phi_2} \ldots a_{\phi_r} \end{Bmatrix}$$

as *identical* with its *equal*,

$$\begin{Bmatrix} a_{\phi_1}, & a_{\phi_2} \ldots a_{\phi_r} \\ a_{\theta_1}, & a_{\theta_2} \ldots a_{\theta_r} \end{Bmatrix}.$$

If, however, we now convene to treat these two forms as distinct, so that in theorem (2)

$$\Sigma \left\{ Q \begin{pmatrix} \theta_1, & \theta_2 \ldots \theta_r \\ \phi_1, & \phi_2 \ldots \phi_r \end{pmatrix} \times \begin{pmatrix} a_{\theta_1}, & a_{\theta_2} \ldots a_{\theta_r} \\ a_{\phi_1}, & a_{\phi_2} \ldots a_{\phi_r} \end{pmatrix} \right\}$$

will contain $\left\{ \dfrac{n(n-1)\ldots(n-r+1)}{1.2\ldots r} \right\}^2$ terms, then we may write simply

$$Q \begin{pmatrix} \theta_1, & \theta_2 \ldots \theta_r \\ \phi_1, & \phi_2 \ldots \phi_r \end{pmatrix} = \begin{Bmatrix} a_{k_1}, & a_{k_2} \ldots a_{k_r} \\ b_{\theta_1}, & b_{\theta_2} \ldots b_{\theta_r} \end{Bmatrix} \times \begin{Bmatrix} a_{l_1}, & a_{l_2} \ldots a_{l_r} \\ b_{\phi_1}, & b_{\phi_2} \ldots b_{\phi_r} \end{Bmatrix},$$

* Perhaps the most remarkable indirect question to which the method of determinants has been hitherto applied is Hesse's problem of reducing a cubic function of 3 letters to another consisting only of 4 terms by linear substitutions—a problem which appears to set at defiance all the processes and artifices of common algebra. I have succeeded in applying a method founded upon this calculus to the linear reduction of a biquadratic function of two letters to Cayley's form $x^4 + mx^2y^2 + y^4$, and of a 5^e function of two letters to the new form $x^5 + y^5 + (ax + by)^5$. This last reduction is effected by means of the properties of a certain other function of the 8th degree connected with the given function of the 5th degree. See a paper on this subject in the forthcoming May Number of the *Cambridge and Dublin Mathematical Journal*. [p. 191 above.]

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which equation is subject to no exception for the case of the θ's and ϕ's becoming identical. As regards this theorem, it will not fail to strike the reader that it ought to admit of verification; for that U may be derived from V in the same manner as V from U if we express $y_1, y_2 \dots y_n$ in terms of $x_1, x_2 \dots x_n$, by solving the system of equations (2), which there is no difficulty in doing. In fact, if we write

$$y_1 = \alpha_1\beta_1 x_1 + \alpha_1\beta_2 x_2 + \dots + \alpha_1\beta_n x_n,$$
$$y_2 = \alpha_2\beta_1 x_1 + \alpha_2\beta_2 x_2 + \dots + \alpha_2\beta_n x_n,$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots$$
$$y_n = \alpha_n\beta_1 x_1 + \alpha_n\beta_2 x_2 + \dots + \alpha_n\beta_n x_n,$$

we shall obtain

$$\alpha_r\beta_s = \begin{Bmatrix} a_1, & a_2 \dots a_{r-1}, & a_{r+1}, & a_{r+2} \dots a_n \\ b_1, & b_2 \dots b_{s-1}, & b_{s+1}, & b_{s+2} \dots b_n \end{Bmatrix} \div \begin{Bmatrix} a_1, & a_2 \dots a_n \\ b_1, & b_2 \dots b_n \end{Bmatrix}.$$

Accordingly we shall find

$$\begin{Bmatrix} a_{m_1}, & a_{m_2} \dots a_{m_r} \\ a_{p_1}, & a_{p_2} \dots a_{p_r} \end{Bmatrix} = \Sigma \left\{ Q \begin{pmatrix} \psi_1, & \psi_2 \dots \psi_r \\ \omega_1, & \omega_2 \dots \omega_r \end{pmatrix} \times \begin{pmatrix} b_{\psi_1}, b_{\psi_2} \dots b_{\psi_r} \\ b_{\omega_1}, b_{\omega_2} \dots b_{\omega_r} \end{pmatrix} \right\},$$

and

$$Q \begin{pmatrix} \psi_1, \psi_2 \dots \psi_r \\ \omega_1, & \omega_2 \dots \omega_r \end{pmatrix} = \begin{pmatrix} \alpha_{m_1}, & \alpha_{m_2} \dots \alpha_{m_r} \\ \beta_{\psi_1}, & \beta_{\psi_2} \dots \beta_{\psi_r} \end{pmatrix} \times \begin{pmatrix} \alpha_{p_1}, & \alpha_{p_2} \dots \alpha_{p_r} \\ \beta_{\omega_1}, & \beta_{\omega_2} \dots \beta_{\omega_r} \end{pmatrix};$$

substituting for the α's and β's their symbolical equivalents given above, and applying the theorem given below, we shall easily obtain

$$Q \begin{pmatrix} \psi_1, \psi_2 \dots \psi_r \\ \omega_1, & \omega_2 \dots \omega_r \end{pmatrix} = \begin{pmatrix} a_{m_{r+1}}, & a_{m_{r+2}} \dots a_{m_n} \\ b_{\psi_{r+1}}, & b_{\psi_{r+2}} \dots b_{\psi_n} \end{pmatrix} \times \begin{pmatrix} \alpha_{p_{r+1}}, & \alpha_{p_{r+2}} \dots \alpha_{p_n} \\ \beta_{p_{r+1}}, & \beta_{p_{r+2}} \dots \beta_{p_n} \end{pmatrix}$$
$$\div \begin{Bmatrix} a_1, a_2 \dots a_n \\ b_1, b_2 \dots b_n \end{Bmatrix}^2.$$

If, now, in the expression

$$\begin{Bmatrix} b_{k_1}, & b_{k_2} \dots b_{k_r} \\ b_{l_1}, & b_{l_2} \dots b_{l_r} \end{Bmatrix} = \Sigma \left\{ \begin{pmatrix} a_{k_1}, & a_{k_2} \dots a_{k_r} \\ b_{\theta_1}, & b_{\theta_2} \dots b_{\theta_r} \end{pmatrix} \begin{pmatrix} a_{l_1}, & a_{l_2} \dots a_{l_r} \\ b_{\phi_1}, & b_{\phi_2} \dots b_{\phi_r} \end{pmatrix} \begin{pmatrix} a_{\theta_1}, & a_{\theta_2} \dots a_{\theta_r} \\ a_{\phi_1}, & a_{\phi_2} \dots a_{\phi_r} \end{pmatrix} \right\},$$

we resubstitute for $\begin{Bmatrix} a_{\theta_1}, a_{\theta_2} \dots a_{\theta_r} \\ a_{\phi_1}, a_{\phi_2} \dots a_{\phi_r} \end{Bmatrix}$ its value in the form of

$$\Sigma \left\{ \begin{pmatrix} b_{\omega_1}, b_{\omega_2} \dots b_{\omega_r} \\ b_{\psi_1}, b_{\psi_2} \dots b_{\psi_r} \end{pmatrix} Q \right\},$$

we shall obtain $\begin{pmatrix} b_{k_1}, b_{k_2} \dots b_{k_r} \\ b_{l_1}, b_{l_2} \dots b_{l_r} \end{pmatrix}$ under the form of

$$\Sigma \left\{ R \begin{pmatrix} \omega_1, & \omega_2 \dots \omega_r \\ \psi_1, & \psi_2 \dots \psi_r \end{pmatrix} \times \begin{pmatrix} b_{\omega_1}, b_{\omega_2} \dots b_{\omega_r} \\ b_{\psi_1}, b_{\psi_2} \dots b_{\psi_r} \end{pmatrix} \right\};$$

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and $R\begin{pmatrix}\omega_1, \omega_2 \dots \omega_r\\ \psi_1, \psi_2 \dots \psi_r\end{pmatrix}$ must $=0$, except for the case of $\omega_1, \omega_2 \dots \omega_r$; $\psi_1, \psi_2 \dots \psi_r$ being respectively identical with $k_1, k_2 \dots k_r$; $l_1, l_2 \dots l_r$, for which case $R\begin{pmatrix}k_1, k_2 \dots k_r\\ l_1, l_2 \dots l_r\end{pmatrix}$ must be unity. I have gone through this calculation and verified the result; in order to effect which, however, the following important generalization of theorem (1) must be apprehended.

Suppose two sets of umbræ,

$$a_1, a_2 \dots a_{m+n},$$

$$b_1, b_2 \dots b_{m+n},$$

and let r be any number less than m, and let any r-ary combination of the m numbers 1, 2, 3 ... m be expressed by ${}^q\theta_1, {}^q\theta_2 \dots {}^q\theta_m$, where q goes through all the values intermediate between 1 and μ, μ being

$$\frac{m(m-1)\dots(m-r+1)}{1.2\dots r};$$

then I say that the compound determinant,

$$\overline{\begin{matrix} a_{{}^1\theta_1}, & a_{{}^1\theta_2} \dots a_{{}^1\theta_m}, & a_{m+1}, & a_{m+2} \dots a_{m+n} \\ b_{{}^1\theta_1}, & b_{{}^1\theta_2} \dots b_{{}^1\theta_m}, & b_{m+1}, & b_{m+2} \dots b_{m+n} \end{matrix}\Big|} \quad \overline{\begin{matrix} a_{{}^2\theta_1}, & a_{{}^2\theta_2} \dots a_{{}^2\theta_m}, & a_{m+1}, & a_{m+2} \dots a_{m+n} \\ b_{{}^2\theta_1}, & b_{{}^2\theta_2} \dots b_{{}^2\theta_m}, & b_{m+1}, & b_{m+2} \dots b_{m+n} \end{matrix}\Big|}$$

$$\dots\dots \overline{\begin{matrix} a_{{}^\mu\theta_1}, & a_{{}^\mu\theta_2} \dots a_{{}^\mu\theta_m}, & a_{m+1}, & a_{m+2} \dots a_{m+n} \\ b_{{}^\mu\theta_1}, & b_{{}^\mu\theta_2} \dots b_{{}^\mu\theta_m}, & b_{m+1}, & b_{m+2} \dots b_{m+n}, \end{matrix}\Big|}$$

is equal to the following product,

$$\overline{\begin{matrix} a_{m+1}, & a_{m+2} \dots a_{m+n} \\ b_{m+1}, & b_{m+2} \dots b_{m+n} \end{matrix}\Big|}^{\mu'}, \quad \overline{\begin{matrix} a_1, & a_2 \dots a_{m+n} \\ b_1, & b_2 \dots b_{m+n} \end{matrix}\Big|}^{\mu''} \tag{4}$$

where

$$\mu'' = \frac{(m-1)(m-2)\dots(m-r+1)}{1.2\dots(r-1)},$$

and

$$\mu' = \frac{(m-1)(m-2)\dots(m-r)}{1.2\dots r};$$

when $r=1$, we have the case already given in theorem (2), and of course μ'' is to be taken unity.

This very general theorem is itself several degrees removed from my still unpublished Fundamental Theorem which is a theorem for the expansion of the products of determinants.

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Obs. The analogy upon which the extension of the Vandermondian notation from simple to compound determinants is grounded, would be better apprehended if the biliteral symbols of simple quantities were written with the umbral elements disposed vertically, as $\begin{matrix} a \\ b \end{matrix}$, instead of horizontally, as ab; which latter is the method for the purposes of typographical uniformity adopted in the text above. The other mode is, however, much to be preferred, and is what I propose hereafter to adhere to. For my two general umbræ, a, b, Vandermonde uses two *numbers*, one set a-cock upon the other, as 5^4. The objection to the use of numbers is apparent as soon as it becomes necessary to treat of the mutual relations of diverse systems of determinants, and his mode of writing the umbræ militates against the perception of the most valuable algebraical analogies. The one important point in which Vandermonde has anticipated me, consists in expressing a simple determinant by two horizontal rows of umbræ one over the other. But the idea upon which this depends is so simple and natural, that it was sure to reappear in any well-constructed system of notation.

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