53.

ON THE RELATION BETWEEN THE VOLUME OF A TETRAHEDRON AND THE PRODUCT OF THE *SIXTEEN* ALGEBRAICAL VALUES OF ITS SUPERFICIES.

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THE area of a triangle is related (as is well known) in a very simple manner to the eight algebraical values of its perimeter: If we call the values of the squared sides of the triangle a, b, c, there will be nothing to distinguish the algebraical affections of sign of the simple lengths so as to entitle one to a preference over the other. The area of the triangle can only vanish by reason of the three vertices coming into a straight line; hence, according to the general doctrine of characteristics, we must have the Norm of $\sqrt{a}+\sqrt{b}+\sqrt{c}$, containing as a factor some root or power of the expressions for the area of the triangle. The Norm in question being representable as $-N^2$ where N is the Norm of $a^{\frac{1}{2}} \pm b^{\frac{1}{2}} \pm c^{\frac{1}{2}}$, which is of four dimensions in the elements a, b, c, and undecomposable into rational factors, we infer that to a numerical factor *près* the square of the area must be identical with the Norm N, and thus, by a logical *coup-de-main*, completely supersede all occasion for the ordinary geometrical demonstration given of this proposition, which in its turn, with certain superadded definitions, would admit of being adopted as the basis of an absolutely pure system of Analytical Trigonometry that should borrow nothing from the methods and results of sensuous or practical geometry. But into this speculation it is not my present purpose to enter: what I propose to do is to extend a similar mode of reasoning to space of three dimensions, and to point out a general theorem in determinants which is involved as a consequence in the generalization of the result of the inquiry when pushed forward into the regions of what may be termed Absolute or Universal Rational Space.

Let F, G, H, K be the four squared areas of the faces of a tetrahedron, and V the volume; then, since V only becomes zero in the case of the four vertices coming into the same plane, which is characterised by the equation

$$\sqrt{F}+\sqrt{G}+\sqrt{H}+\sqrt{K}=0$$

subsisting, we infer that N the Norm of

$$\sqrt{F} \pm \sqrt{G} \pm \sqrt{H} \pm \sqrt{K}$$

must contain a power of V as a rational factor. V^2 is rational and of three dimensions in the squared edges; the Norm above spoken of is of eight dimensions in the same. Consequently there is a rational factor, say Q, remaining, which is of five dimensions in the squared edges, and this factor I now proceed to determine, the other factor V^2 being, as is well known, a numerical product of the determinant

$$\begin{vmatrix} 0, & ab^2, & ac^2, & ad^2, & 1 \\ ba^2, & 0, & bc^2, & bd^2, & 1 \\ ca^2, & cb^2, & 0, & cd^2, & 1 \\ da^2, & db^2, & dc^2, & 0, & 1 \\ 1, & 1, & 1, & 1, & 0 \end{vmatrix}$$

a, b, c, d being the four angular points of the tetrahedron. See *London and Edinburgh Philosophical Magazine*, 1852. [p. 386 above.]

The quantity Q possesses an interest of a geometrical character; for if we call the radii of the eight spheres which can be inscribed in a tetrahedron $r_1, r_2, r_3, r_4, r_5, r_6, r_7, r_8$, we evidently have $r_1 r_2 r_3 r_4 r_5 r_6 r_7 r_8 \times N = (3V)^8$. Hence (R), the product of the eight radii in question, $= \dfrac{3^8 V^8}{N} = \dfrac{3^8 V^6}{Q}$.

Consequently Q is the quantity which characterises the fact of one or more of *the radii of the inscribed spheres becoming infinite*. For the triangle there exists no corresponding property; this we know *à priori*, and can explain also analytically from the fact that if we call P the product of the radii of the four inscribable circles, ν the Norm of the perimeter, and A the area, we have

$$P\nu = 2^4 A^4,$$

and

$$\nu = \frac{2^4 A^4}{P} = A^2,$$

which contains no denominator capable of becoming zero, so that as long as the sides remain finite the curvature of the inscribed circles is incapable of vanishing.

To determine N as a function of the edges, and then to discover by actual division the value of $\dfrac{N}{V^2}$, would be the direct but an excessively tedious and almost impracticably difficult process. I have ever felt a preference for the *à priori* method of discovering forms whose properties are known, and never yet have met with an instance where analysis has denied to gentle solicitation conclusions which she would be loth to grant to the application of force. The case before us offers no exception to the truth of this remark. Q is a function of five dimensions in terms of the squared edges: let us begin by finding the value of that part of Q in which at most a certain set of four of these edges make their appearance, and to find which consequently the other two edges may be supposed zero without affecting the result. We may make two distinct hypotheses concerning these two edges; we may suppose that they are opposite, that is non-intersecting edges, or that they are contiguous, that is intersecting edges.

To meet the first hypothesis suppose $ab = 0$, $ce = 0$.

For convenience sake, use F, G, H, K to denote 16 times the square of each area, instead of the simple square of the areas. Call

$$16\,(abc)^2 = K, \quad 16\,(abd)^2 = H, \quad 16\,(acd)^2 = G, \quad 16\,(bcd)^2 = F.$$

Then

$$-K = (ab)^4 + (ac)^4 + (bc)^4 - 2\,(ab)^2\,(ac)^2 - 2\,(ab)^2\,(bc)^2 - 2\,(ac)^2\,(bc)^2$$
$$= ac^4 + bc^4 - 2\,(ac)^2\,(bc)^2.$$

Similarly,

$$-H = ad^4 + bd^4 - 2\,(ad)^2\,(bd)^2,$$
$$-G = ca^4 + da^4 - 2ca^2da^2,$$
$$-F = cb^4 + db^4 - 2cb^2db^2.$$

Hence one value of $\sqrt{F} + \sqrt{G} + \sqrt{H} + \sqrt{K}$ will be

$$\sqrt{(-1)}\,\{(ac^2 - bc^2) + (bd^2 - ad^2) + (da^2 - ac^2) + (bc^2 - bd^2)\} = 0.$$

Hence, on this first supposition, the Norm vanishes. But V^2 does not vanish when $ab = 0$, $cd = 0$, for it becomes, *saving* a numerical factor,

$$\begin{vmatrix} 0, & 0, & ac^2, & ad^2, & 1 \\ 0, & 0, & bc^2, & bd^2, & 1 \\ ca^2, & cb^2, & 0, & 0, & 1 \\ da^2, & db^2, & 0, & 0, & 1 \\ 1, & 1, & 1, & 1, & \end{vmatrix},$$

that is

$$(ac^2 . bd^2 - ad^2 . bc^2)\,(cb^2 + ad^2 - ca^2 - bd^2)$$
$$+ (bc^2 - ac^2)\,(ca^2 . db^2 - cb^2 . da^2)$$
$$+ (ad^2 - bd^2)\,(ca^2 . db^2 - cb^2 . da^2)$$
$$= 2\,(ac^2 . bd^2 - ad^2 . bc^2)\,(ad^2 + bc^2 - ac^2 - bd^2);$$

and consequently, since N vanishes but V^2 does not vanish, Q vanishes, showing that there is no term in Q but what contains one at least of any two opposite edges as a factor; or, in other words, there is no term in Q of which the product of the square of the product of all three sides of some one or other of the four faces does not form a constituent part.

Next, let us suppose $ab = 0$, $ac = 0$, then

$$\begin{aligned}
K^2 &= 16abc^2 = -\,bc^4,\\
H^2 &= 16abd^2 = -\,(ad^2 - bd^2)^2,\\
G^2 &= 16acd^2 = -\,(ad^2 - cd^2)^2,\\
F^2 &= 16bcd^2 = -\,bc^4 - bd^4 - cd^4 + 2bc^2 \,.\, bd^2 + 2bc^2 \,.\, cd^2 + 2bd^2 \,.\, cd^2.
\end{aligned}$$

Four of the factors of N will be therefore

$$\{\iota\,(bc^2 + cd^2 - bd^2) \pm F\}, \quad \{\iota\,(bc^2 - cd^2 + bd^2) \pm F\},$$

ι denoting $\surd(-1)$, and the product of these four factors will be

$$\{(bc^2 + cd^2 - bd^2)^2 + F^2\} \times \{(bc^2 - cd^2 + bd^2)^2 + F^2\},$$

which is equal to

$$16bc^4 \,.\, bd^2 \,.\, cd^2;$$

and similarly, the remaining part of the Norm will be

$$\{(2ad^2 - bd^2 - cd^2 + bc^2)^2 + F^2\} \times \{(2ad^2 - bd^2 - cd^2 - bc^2)^2 + F^2\},$$

that is

$$\{4ad^4 - 4ad^2\,(bd^2 + cd^2 + bc^2) + 4bc^2 \,.\, bd^2 + 4bd^2 \,.\, cd^2 + 4cd^2 \,.\, bc^2\}$$
$$\times\,\{4ad^4 - 4ad^2\,(bd^2 + cd^2 - bc^2) + 4bd^2 \,.\, cd^2\}.$$

Again, since $ac^2 = 0$ and $bc^2 = 0$, V^2 becomes

$$\begin{vmatrix}
0, & 0, & 0, & ad^2, & 1\\
0, & 0, & bc^2, & bd^2, & 1\\
0, & cb^2, & 0, & cd^2, & 1\\
da^2, & db^2, & dc^2, & 0, & 1\\
1, & 1, & 1, & 1, & 0
\end{vmatrix},$$

which is evidently equal to

$$2bc^2 \begin{vmatrix}
0, & 0, & ad^2, & 1\\
0, & cb^2, & cd^2, & 1\\
da^2, & db^2, & 0, & 1\\
1, & 1, & 1, &
\end{vmatrix} - bc^4 \begin{vmatrix}
0, & ad^2, & 1\\
da^2, & 0, & 1\\
1, & 0, &
\end{vmatrix},$$

$$\begin{aligned}
&= 2bc^2\,\{2bc^2ad^2 + ad^4 - ad^2bd^2 - cd^2ad^2 + bd^2cd^2\} - 2bc^4ad^2\\
&= 2bc^2\,\{ad^4 - ad^2\,(bd^2 + cd^2 - bc^2) + bd^2 \,.\, cd^2\}.
\end{aligned}$$

Hence, paying no attention to any mere numerical factor, we have found that when $ac = 0$ and $bc = 0$, Q or $\frac{N}{V^2}$ becomes

$$bc^2 . bd^2 . cd^2 \{ad^4 - ad^2 (bd^2 + cd^2 + bc^2) + bc^2 . bd^2 + bd^2 . cd^2 + cd^2 . bc^2\}.$$

Hence, with the exception of the terms in which five out of the six edges enter, the complete value of Q will be

$$\Sigma (bc^2 . bd^2 . cd^2) \{ad^4 - ad^2 (bd^2 + cd^2 + bc^2) + bc^2 . bd^2 + bd^2 . cd^2 + cd^2 . bc^2\},$$

or more fully expressed, and still abstracting from terms containing five edges,

$$= \Sigma bc^2 . bd^2 . cd^2 \{(ab^4 + ac^4 + ad^4) - (ab^2 + ac^2 + bc^2)(bd^2 + bc^2 + cd^2) + bc^2 . bd^2 + bd^2 . cd^2 + cd^2 . bc^2\}.$$

It remains only to determine the value of the numerical coefficient affecting each of the six terms of the form

$$ab^2 . ac^2 . ad^2 . bc^2 . bd^2.$$

To find this, let

$$ab^2 = ac^2 = ad^2 = bc^2 = bd^2 = cd^2 = 1\,;$$

then evidently, since all the squared areas are equal, several of the factors of N will become zero, but V^2 evidently does not become zero for a regular tetrahedron; hence Q becomes zero: and if we call the numerical factor sought for λ, we must have (observing that the Σ includes four parts corresponding to each of the four faces)

$$4\{3 - 9 + 3\} + 6\lambda = 0,$$

therefore $$-12 + 6\lambda = 0, \text{ or } \lambda = 2.$$

Hence the complete value of Q is

$$\Sigma ab^2 . bc^2 . ca^2 \{(da^4 + db^4 + dc^4) - (da^2 + db^2 + dc^2)(ab^2 + bc^2 + ca^2) + ab^2 . bc^2 + bc^2 . ca^2 + ca^2 . ab^2\} + 2\Sigma (ab^2 . bc^2 . cd^2 . da^2 . ac^2)\,;$$

or, which is the same quantity somewhat differently and more simply arranged,

$$Q = \Sigma (ab^2 . bc^2 . ca^2) \{(da^4 + db^4 + dc^4 + da^2 . db^2 + db^2 . dc^2 + dc^2 . da^2) + (ab^2 . bc^2 + bc^2 . ca^2 + ca^2 . ab^2) - (da^2 + db^2 + dc^2)(ab^2 + bc^2 + ca^2)\},$$

and this quantity equated to zero expresses the conditions of a radius of an inscribed sphere becoming infinite. The direct method would have involved, as the first step, the formation of the Norm of a numerator consisting of

$$\sqrt{F} \pm \sqrt{G} \pm \sqrt{H} \pm \sqrt{K},$$

the value of which is

$$\Sigma F^4 - 4\Sigma F^3 G + 6\Sigma F^2 G^2 + 4\Sigma F^2 GH - 40 FGHK,$$

and contains $4 + 6 + 12$, that is 22 positive terms, and 12, that is 13 negative terms, together 35 terms, each of which might be an aggregate of 6^4 or 1296 quantities, and thus involve in all the consideration of 45360 separate parts, for each of the quantities F, G, H, K being a quadratic function of three of the squared edges, will contain six terms. It is not uninteresting to notice that in addition to the case already mentioned of two opposite edges being each zero, as $ab = 0$, $cd = 0$, Q will also vanish for the case of $ab = cd$, $bc = ad$; that is for the case of two intersecting edges being each equal in length to the edges respectively opposite to them. This is evident from the fact that on the hypothesis supposed the face $acb = acd$ and the face $bdc = bda$; hence $N = 0$, and therefore, V not vanishing, $\frac{N}{V^2}$, that is Q, will vanish.

We may moreover remark that since $ab = 0$ and $cd = 0$ does not make V vanish, the perpendicular distance of ab from cd, which, multiplied by $ab \times cd$, gives six times the volumes, must on this supposition become infinite. When three edges lying in the same plane all vanish simultaneously, Q vanishes, since one edge at least in every face of the pyramid vanishes, and V also vanishes, as is evident from the expression for V^2, when $ab = 0$, $ac = 0$, $bc = 0$, becoming a multiple of

$$\begin{vmatrix} 0, & 0, & 0, & ad^2, & 1 \\ 0, & 0, & 0, & bd^2, & 1 \\ 0, & 0, & 0, & cd^2, & 1 \\ ad^2, & bd^2, & cd^2, & 0, & 0 \\ 1, & 1, & 1, & 0, & 0 \end{vmatrix},$$

which is evidently zero.

It appeared to me not unlikely, from the situation and look of Q (the characteristic of one of the inscribed spheres becoming infinite), that it might admit of being represented as a determinant, but I have not succeeded in throwing it under that form. I have a strong suspicion that if we take Q' a function corresponding to a tetrahedron $a'b'c'd'$, in the same way as Q corresponds to $abcd$, QQ', and not improbably $\sqrt{(QQ')}$, will be found to be (as we know from Staudt's Theorem of $\sqrt{(V^2 . V'^2)}$) a rational integral function of the squares of the distances of the points a, b, c, d from the points a', b', c', d'.

That N should divide out by V^2 is in itself an analytical theorem relating to 6 arbitrary quantities $ab^2, ac^2, ad^2, bc^2, bd^2, cd^2$, which evidently admits of extension to any triangular number 10, 15, &c. of arbitrary quantities. Thus we may affirm, *à priori*, that the norm of

$$\sqrt{L} \pm \sqrt{M} \pm \sqrt{N} \pm \sqrt{P} \pm \sqrt{Q},$$

where (for the sake of symmetry, retaining double letters, as AB, AC, &c., to denote *simple* quantities)

$$Q = \begin{vmatrix} 0, & AB, & AC, & AD, & 1 \\ AB, & 0, & BC, & BD, & 1 \\ AC, & BC, & 0, & CD, & 1 \\ AD, & BD, & CD, & 0, & 1 \\ 1, & 1, & 1, & 1, & 0 \end{vmatrix}, \quad P = \begin{vmatrix} 0, & AB, & AC, & AE, & 1 \\ AB, & 0, & BC, & BE, & 1 \\ AC, & BC, & 0, & CE, & 1 \\ AE, & BE, & CE, & 0, & 1 \\ 1, & 1, & 1, & 1, & 0 \end{vmatrix},$$

$$N = \&c., \quad M = \&c., \quad L = \&c.,$$

will contain as a factor the determinant

$$\begin{vmatrix} 0, & AB, & AC, & AD, & AE, & 1 \\ AB, & 0, & BC, & BD, & BE, & 1 \\ AC, & BC, & 0, & CD, & CE, & 1 \\ AD, & BD, & CD, & 0, & DE, & 1 \\ AE, & BE, & CE, & DE, & 0, & 1 \\ 1, & 1, & 1, & 1, & 1, & 0 \end{vmatrix},$$

and a similar theorem may evidently be extended to the case of any $\frac{n(n+1)}{2}$ arbitrary quantities whatever.