64.

NOTE ON THE NEW RULE OF LIMITS.

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IT may appear like harping too long on the same string to add any further remarks on the rule relating to so simple and elementary a matter as that of assigning limits to the roots of a given algebraical equation; but it will be remembered that some of the greatest masters of analysis, including the honoured names of Newton and Cauchy, have not disdained to treat, and to give to the world their comparatively imperfect results on this very subject. I hope, therefore, to stand excused of any undue egotism in adding some observations which may tend to present, under a clearer aspect and more finished form, the new and beautifully flexible rule laid before the readers of this *Magazine* in the two preceding Numbers.

Firstly, I observe that any succession of signs may be considered as made up of, and decomposable into, sequences of changes exclusively, if we agree to consider, where necessary, a single isolated sign $+$ or $-$ as a sequence of zero changes. Thus, for instance, $+--++++-+++-+--$ may be treated as made up of the variation sequences

$$+-,\ -+,\ +,\ +,\ +-+,\ +,\ +-+-,\ -*.$$

Secondly, I observe that if $X_1, X_2 \ldots X_i$ be all linear functions of x, and the signs of the coefficients of x in these functions constitute a single unbroken series of variations, the denominator of the continued fraction

$$\frac{1}{X_1+}\,\frac{1}{X_2+}\,\frac{1}{X_3+}\cdots\frac{1}{X_i}$$

(reduced to the form of an ordinary algebraical fraction) will have all its roots real.

* The rule is, that the given series of signs is to be separated into distinct sequences of variations, so that the final term of one sequence and the initial term of the next shall form a continuation, that is we must have variation sequences connected together by continuations at their joinings.

Thirdly, suppose, for greater simplicity, that ϕx is of one degree in x lower than fx, and that by the ordinary process of common measure we obtain

$$\frac{\phi x}{fx} = \frac{1}{X_1 +} \frac{1}{X_2 +} \frac{1}{X_3 +} \cdots \frac{1}{X_n},$$

where $X_1, X_2, X_3 \ldots X_n$ are all of them linear functions of x.

Let $X_1, X_2 \ldots X_n$ be divided into distinct and unblending sequences,

$$X_1 X_2 \ldots X_i, \quad X_{i+1} X_{i+2} \ldots X_{i'}, \quad X_{i'+1} \ldots X_{i''}, \quad \ldots, \quad X_{(i)+1} X_{(i)+2} \ldots X_n;$$

so that in each sequence the signs of the coefficients of x present a single unbroken series of variations, which by virtue of observation (1), may be considered to be always capable of being done, and let

$$\frac{\phi_1 x}{f_1 x} = \frac{1}{X_1 +} \frac{1}{X_2 +} \frac{1}{X_3 +} \cdots \frac{1}{X_i},$$

$$\frac{\phi_2 x}{f_2 x} = \frac{1}{X_{i+1} +} \frac{1}{X_{i+2}} \cdots\cdots \frac{1}{X_{i'}},$$

....

....

$$\frac{(\phi)\, x}{(f)\, x} = \frac{1}{X_{(i)+1} +} \cdots\cdots\cdots\cdots \frac{1}{X_n};$$

then, according to observation (2), the equations

$$f_1 x = 0, \quad f_2 x = 0 \ldots (f)\, x = 0,$$

have each of them all their roots real; and the observation now to be made is, that the highest of the highest roots and the lowest of the lowest roots of these equations furnish respectively a superior and inferior limit to the roots of $fx = 0$*.

* This theorem may be more concisely stated as follows:—"If U with any subscript be understood to mean a linear function of x in which the sign of the coefficient of x is constant, then the finite roots of the equation

$$\frac{1}{U_1 -} \frac{1}{U_2 -} \frac{1}{U_3 -} \cdots \frac{1}{U_i +} \frac{1}{U_{i+1} -} \frac{1}{U_{i+2} -} \cdots \frac{1}{U_{i'} +} \cdots \frac{1}{U_{(i)+1} -} \frac{1}{U_{(i)+2} -} \cdots \frac{1}{U_n} = \infty$$

lie between the greatest and least finite roots of the equations

$$\frac{1}{U_1 -} \frac{1}{U_2 -} \cdots \frac{1}{U_i} = \infty,$$

$$\frac{1}{U_{i+1} -} \frac{1}{U_{i+2} -} \cdots \frac{1}{U_{i'}} = \infty,$$

....

....

$$\frac{1}{U_{(i)} -} \frac{1}{U_{(i)+1} -} \cdots \frac{1}{U_n} = \infty."$$

The theorem under this form suggests a much more general one relating to para-symmetrical determinants, that is determinants partly normal and partly *gauche*, which will be given hereafter; one example among the many confirming the importance of the view first stated in this *Magazine* by the author of this paper, whereby continued fractions are incorporated with the doctrine of determinants.

N.B. The single root of any one or more of these which may be of the first degree in x is to be treated, in applying the preceding observation, as being at the same time the highest and the lowest root of such equation or equations.

Fourthly and lastly, the problem of assigning limits to the roots of $fx = 0$ reduces itself to that of finding limits to

$$f_1x = 0, \quad f_2x = 0 \ldots (f)\, x = 0;$$

for the greatest and least of these collectively will evidently, *à fortiori*, by virtue of the preceding observation, be limits to the roots of $fx = 0$. Of any such of these as are linear, the root or roots themselves may be treated as known; leaving these out of consideration, the functional part of any other of them, such as f_1x, is the denominator of a continued fraction of the form

$$\frac{1}{(a_1x+b_1)+}\,\frac{1}{(a_2x+b_2)+}\,\frac{1}{(a_3x+b_3)+}\cdots\frac{1}{(a_ix+b_i)},$$

in which $a_1, a_2, a_3 \ldots a_i$ present a single sequence of variations of sign, and the limits to the roots of $f_1x = 0$ may be found as follows.

Form the two systems of equations (in which $\mu_1, \mu_2 \ldots \mu_{i-1}$ are numerical quantities having all the same algebraical sign, but are otherwise arbitrary and independent),

$$\begin{array}{rl|rl}
a_1x + b_1 = & \mu_1 & a_1x + b_1 = & -\mu_1 \\
a_2x + b_2 = & -\mu_2 - \dfrac{1}{\mu_1} & a_2x + b_2 = & \mu_2 + \dfrac{1}{\mu_1} \\
a_3x + b_3 = & \mu_3 + \dfrac{1}{\mu_2} & a_3x + b_3 = & -\mu_3 - \dfrac{1}{\mu_2} \\
\cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots \\
a_{i-1}x + b_{i-1} = & (-)^{i-2}\mu_{i-1} + \dfrac{(-)^{i-2}}{\mu_{i-2}} & a_{i-1}x + b_{i-1} = & (-)^{i-1}\mu_{i-1} + \dfrac{(-)^{i-1}}{\mu_{i-2}} \\
a_ix + b_i = & \dfrac{(-)^{i-1}}{\mu_{i-1}} & a_ix + b_i = & \dfrac{(-)^{i}}{\mu_{i-1}}
\end{array}$$

then (supposing μ_1 to have the same sign as a_1) the highest of the values of x obtained from the first system, and the lowest of the values of x found from the second system of these equations, will be a superior and inferior limit respectively to the roots of $f_1x = 0$; and so for all the rest of the equations

$$f_2(x) = 0, \; f_3(x) = 0 \ldots (f)\, x = 0,$$

excluding those of the first degree.

It will be seen that the theorems contained in the observations (3) and (4) combined (which presuppose the statements made in observations (1) and (2)), contain between them the theorem given in the last Number of the *Magazine* [p. 627 above], but rendered in one or two particulars more simple and precise, and, as it were, reduced to its lowest terms. In the whole course of my experience I never remember a theory which has undergone so many successive transformations in my mind as this very simple one, since the day when I first unexpectedly discovered the germ of it in results obtained for quite a different purpose. In fact, it never entered into my thoughts that in so beaten a track, and in so hackneyed a subject as that of finding numerical limits to the roots of an equation, there was left anything to be discovered; and my sole merit, if any, in bringing the new rule to light, consists in having been able to detect the presence and appreciate the value of a truth which fortune or providence had put into my hands.