66.

ON THE EXPLICIT VALUES OF STURM'S QUOTIENTS.

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By Sturm's quotients is of course meant to be understood the quotients which result from applying the process for the discovery of the greatest common measure between fx (an algebraical function of the nth degree in x, and whose first coefficient is unity) and $f'x$ its first derivative, as in Sturm's theorem; or which is the same thing in effect, supposing $\frac{f'x}{fx}$ to be represented by

$$\frac{1}{Q_1 -}\,\frac{1}{Q_2 -}\,\frac{1}{Q_3 -}\cdots\frac{1}{Q_n},$$

(where $Q_1, Q_2 \ldots Q_n$ are all linear functions of x), the quotients in question are $Q_1, Q_2 \ldots Q_n$. Before proceeding to discuss these quotients, it will be well to state the form under which the other quantities which appear in the course of the application of the Sturmian process admit of being represented. First, then, it will be remembered that the residues with the signs changed are all of the form

$$R_i = M_i \Sigma \{\zeta(h_1, h_2 \ldots h_i)(x - h_{i+1})(x - h_{i+2}) \ldots (x - h_n)\},$$

where $\zeta(h_1, h_2 \ldots h_i)$ indicates the squared differences between every two of the quantities $h_1, h_2 \ldots h_i$, and $h_1, h_2 \ldots h_n$ are supposed to be the n roots of fx; and where, using ζ_i to denote $\Sigma\zeta(h_1, h_2 \ldots h_i)$, with the convention that $\zeta_0 = 1$, $\zeta_1 = n$, and understanding by (i), $\frac{1}{2}\{1 + (-)^i\}$,

$$M_i = \frac{\zeta^2_{i-2}\,\zeta^2_{i-4} \cdots \zeta^2_{(i)+1}}{\zeta^2_{i-1}\,\zeta^2_{i-3} \cdots \zeta^2_{(i)}}.$$

Here it will be observed that the only quantities appearing are the factors and the differences of the roots of fx; and since these latter are the same as the differences between the corresponding factors, for

$$(x - h) - (x - h') = h' - h,$$

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the entire quantity which expresses any residue R_i may be considered as a function *of the factors of fx exclusively*.

Again, if we solve the syzygetic equation

$$N_i fx + D_i f'x = R_i,$$

I have published many years ago in this *Magazine* the value of D_i, and subsequently in a paper read before the Royal Society on the 16th of June last [p. 429 above] the value of N_i, both which values are also functions of the factors of fx exclusively. $\frac{N_i}{D_i}$, it is easily seen, represents the successive convergents to the continued fraction by which $\frac{f'x}{fx}$ is supposed to be expressed, and R_i (to a constant factor *près*) is the denominator of the reverse convergents of the same continued fraction. To the completion of this part of the theory it evidently therefore becomes necessary to express the quotients $Q_1, Q_2, Q_3 \ldots Q_{n-1}, Q_n$ (of which the first $(n-1)$ are those which appear in Sturm's process, and the last is simply the penultimate Sturmian residue divided by the ultimate residue) under a similar form, that is as functions exclusively of the factors of fx, or, which comes to the same thing, of the factors and the differences of the roots. Guided by an instinctive sense of the beautiful and fitting, in a happy moment I have succeeded in grasping this much wished for representation, with which I propose now and for ever to take my farewell of this long and deeply excogitated theorem.

If we write [cf. p. 499 above, and the Author's footnote, p. 495]

$$R_{i-1} = M_{i-1}\{A_{i-1}x^{n-i+1} - B_{i-1}x^{n-i} + \&c.\},$$

and

$$R_i = M_i\{A_i x^{n-i} - B_i x^{n-i-1} + \&c.\},$$

we have

$$A_{i-1} = \Sigma\zeta(h_1, h_2 \ldots h_{i-1}), \quad B_{i-1} = \Sigma(h_i + h_{i+1} + \ldots + h_n)\zeta(h_1, h_2 \ldots h_{i-1}),$$

$$A_i = \Sigma\zeta(h_1, h_2 \ldots h_i), \qquad B_i = \Sigma(h_{i+1} + h_{i+2} + \ldots + h_n)\zeta(h_1, h_2 \ldots h_i),$$

and the ith quotient is evidently

$$\frac{M_{i-1}}{M_i}\frac{A_{i-1}A_i x + (A_{i-1}B_i - A_i B_{i-1})}{A_i^2};$$

and this is the quantity (unpromising enough in aspect) to be transformed in the manner prescribed.

M_{i-1}, M_i, and A_i are already given under that form, and I find that, putting

$$T_i = A_{i-1}A_i x + (A_{i-1}B_i - A_i B_{i-1}),$$

T_i may be represented by the double sum

$$\Sigma\{[\Sigma\{\zeta(h_{\theta_1}, h_{\theta_2} \ldots h_{\theta_{i-1}})(h_1 - h_{\theta_1})(h_1 - h_{\theta_2}) \ldots (h_1 - h_{\theta_{i-1}})\}]^2 (x - h_1)\}.$$

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This of course implies the truth of the identity

$$\Sigma\{\Sigma\zeta(h_{\theta_1}, h_{\theta_2} \dots h_{\theta_{i-1}})(h_1 - h_{\theta_1})(h_1 - h_{\theta_2}) \dots (h_1 - h_{\theta_{i-1}})\}^2 = A_{i-1} A_i = \zeta_{i-1}\zeta_i,$$

in itself a truly remarkable equation, which it will be seen is of $2(i-1)^2$ dimensions in respect of the roots*.

When $i = 1$,

$$T_1 = \Sigma(x - h_1);$$

and when $i = 2$,

$$T_2 = \Sigma\{[\Sigma(h_1 - h_\theta)]^2 (x - h_1)\},$$

that is

$$= \Sigma\{[(n-1)h_1 - (h_2 + h_3 + \dots + h_n)]^2 (x - h_1)\}.$$

When $i = n$, T_n becomes

$$\Sigma\{\zeta(h_1, h_2 \dots h_n) \times \zeta(h_2, h_3 \dots h_n)(x - h_1)\} = \zeta_n + \Sigma\{\zeta(h_2, h_3 \dots h_n)(x - h_1)\},$$

as it evidently ought to do. Substituting for T_{i-1}, T_i and A_i, their values, we have as the complete general expression of the ith Sturmian quotient the following expression, in which, agreeable to a notation which I have previously used and explained,

$$\begin{bmatrix} h_1 \\ h_{\theta_1}, h_{\theta_2} \dots h_{\theta_{i-1}} \end{bmatrix} \text{ means } (h_1 - h_{\theta_1})(h_1 - h_{\theta_2}) \dots (h_1 - h_{\theta_{i-1}}),$$

namely

$$Q_i = \frac{\zeta^2_{i-1}}{\zeta_i^2} \frac{\zeta^4_{i-3}\zeta^4_{i-5} \dots \zeta^4_{(i)}\dagger}{\zeta^4_{i-2}\zeta^4_{i-4} \dots \zeta^4_{(i)+1}}$$

$$\times \Sigma\left(\left\{\Sigma\left(\zeta(h_{\theta_1}, h_{\theta_2} \dots h_{\theta_{i-1}})\begin{bmatrix} h_1 \\ h_{\theta_1}, h_{\theta_2} \dots h_{\theta_{i-1}} \end{bmatrix}\right)\right\}^2 (x - h_1)\right).$$

It ought not to be passed over in silence, that if we write

$$\frac{1}{Q_1 -}\frac{1}{Q_2 -}\frac{1}{Q_3 -} \dots \frac{1}{Q_i} = \frac{N_i(x)}{D_i(x)};$$

and if we suppose $N_i(x)$ and $D_i(x)$ to be expressed integrally, and to be algebraically prime to one another, then

$$D_{i-1}(x) = \Sigma\left\{\zeta(h_{\theta_1}, h_{\theta_2} \dots h_{\theta_{i-1}})\begin{bmatrix} x \\ h_{\theta_1}, h_{\theta_2} \dots h_{\theta_{i-1}} \end{bmatrix}\right\}.$$

* Thus if $n = 4$ and $i = 2$

$$\zeta_{i-1} = 4, \quad \zeta_2 = \Sigma(h_1 - h_2)^2,$$

and we have

$$4\{(h_1 - h_2)^2 + (h_1 - h_3)^2 + (h_1 - h_4)^2 + (h_2 - h_3)^2 + (h_2 - h_4)^2 + (h_3 - h_4)^2\}$$
$$= (3h_1 - h_2 - h_3 - h_4)^2 + (3h_2 - h_1 - h_3 - h_4)^2 + (3h_3 - h_1 - h_2 - h_4)^2 + (3h_4 - h_1 - h_2 - h_3)^2,$$

and so in general $\zeta_{i-1}\zeta_i$, which is the product of two sums of variable numbers of squares, is expressible rationally as the sum of a constant number (n) of squares for all values of i.

† (i) denotes $\frac{1}{2}\{(-1)^i + 1\}$.

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Hence Q_i is contained as a factor in

$$(D_{i-1}h_1)^2(x-h_1)+(D_{i-1}h_2)^2(x-h_2)\ldots+(D_{i-1}h_n)^2(x-h_n).$$

It may be observed also, that for all values of i between 1 and n inclusively,

$$D_ih_1+D_ih_2+D_ih_3+\ldots+D_ih_n=0,$$

and also that the determinant

$$\begin{vmatrix} 1, & 1, & 1, & \ldots & 1 \\ (D_1h_1)^2, & (D_1h_2)^2, & (D_1h_3)^2 & \ldots & (D_1h_n)^2 \\ (D_2h_1)^2, & (D_2h_2)^2, & (D_2h_3)^2 & \ldots & (D_2h_n)^2 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ (D_{n-1}h_1)^2, & (D_{n-1}h_2)^2, & (D_{n-1}h_3)^2 & \ldots & (D_{n-1}h_n)^2 \end{vmatrix}$$

is always zero [cf. p. 502 above]. To complete the theory, I subjoin the value of N_i, the simplified numerator of the ith convergent to $\frac{f'x}{fx}$, expressed as an improper continued fraction.

Let the sum of the products of $x-h,\ x-k\ldots x-l$ combined i and i together be denoted by $S_i(h,\ k\ldots l)$, and the sum of the ith powers of the same by $\sigma_i(h,\ k\ldots l)$, then N_i is equal to

$$\Sigma\zeta(h_{\theta_1},h_{\theta_2}\ldots h_{\theta_i})\times\{\sigma_{i-1}(h_{\theta_1},h_{\theta_2}\ldots h_{\theta_i})-\sigma_{i-2}(h_{\theta_1},h_{\theta_2}\ldots h_{\theta_i})S_1(h_{\theta_{i+1}}\ldots h_{\theta_n})$$
$$+\sigma_{i-3}(h_{\theta_1},h_{\theta_2}\ldots h_{\theta_i})S_2(h_{\theta_{i+1}}\ldots h_{\theta_n})\mp\&c.$$
$$\ldots\pm(i+1)S_{i-1}(h_{\theta_{i+1}}\ldots h_{\theta_n})\}.$$

The anomaly of the last term being of the form $(1+\sigma_0)S_{i-1}$ (for of course $\sigma_0=i$), instead of being σ_0S_{i-1}, is not a little remarkable.

Of the four sets of Sturmian quantities, namely the residues, the quotients, and the denominators and numerators of the convergents to $\frac{f'x}{fx}$, it will have been seen that the first and third are expressible in terms of the roots and factors by single summations of equal simplicity, the second and fourth by double summations, whereof that which corresponds to the numerators is much the more complicated of the two.

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