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ON A GENERALIZATION OF THE LAGRANGIAN THEOREM OF INTERPOLATION.

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THERE is a well-known theorem of Lagrange for determining the form of a rational integral function of one variable of the degree m, when its values corresponding to $m+1$ values of the variable are assigned. M. Cauchy, in his *Cours d'Analyse de l'École Polytechnique*, has extended this theorem to the case of a rational fraction, of which values corresponding to a sufficient number of values of the variable are given; but the solution of the question there given, although of course correct, is unsatisfactory, as it presents the numerator and denominator under forms not strictly analogous.

The theorem of Lagrange, in respect of its subject matter, may be best generalized as follows.

Suppose any number of rational integral functions of x of the several degrees m_1-1, m_2-1 ... m_i-1, say U_1, U_2 ... U_i, and that the equation

$$l_1U_1+l_2U_2+\ldots+l_iU_i=0$$

is known to be satisfied for $m_1+m_2+\ldots+m_i-1$ (say) $\mu-1$ assigned values of the system of quantities l_1, l_2 ... l_i, x; there will then be $\mu-1$ linear equations connecting the μ coefficients comprised in U_1, U_2 ... U_i, and therefore the ratios of these coefficients, and consequently of the functions to one another, may be determined. There is no difficulty in representing, by aid of the method of determinants, the result of solving these equations whatever be the number of functions; but for the sake of greater simplicity, I shall suppose three only of the several degrees, $e-1$, $i-1$, $\omega-1$ in x, which I shall call U, V, W. Now suppose that $lU+mV+nW=0$ is known to be satisfied for $l=l_t$, $m=m_t$, $n=n_t$, $x=x_t$, t taking all possible values from 1 to $e+i+\omega-1$, say $\tau-1$; let the indices $1, 2, 3 \ldots \tau-1$ be partitioned in every possible way into three groups, containing [when it is the function U which is to be determined] respectively $e-1$, i and ω indices, as

$$\theta_1, \theta_2 \ldots \theta_{e-1};\ \ \theta_e, \theta_{e+1} \ldots \theta_{e+i-1};\ \ \theta_{e+i} \ldots \theta_{\tau-1}$$

(the terms in any group may be arranged indifferently in any order, but are not to be permuted). Let $\zeta^{\frac{1}{2}}(p, q, r \dots s)$ denote in general

$$\begin{aligned}(p-q)\times(p-r)\dots\times(p-s)\\ \times(q-r)\dots\times(q-s)\\ \dots\dots\dots\dots\dots\dots\\ \times(r-s),\end{aligned}$$

and write

$$K_1 = \Sigma(?)\begin{Bmatrix} l_{\theta_1}\dots l_{\theta_{e-1}};\ m_{\theta_2}\dots m_{\theta_{e+i-1}};\ n_{\theta_{e+i}}\dots n_{\theta_{\tau-1}} \\ \zeta^{\frac{1}{2}}(x, x_{\theta_1}\dots x_{\theta_{e-1}})\,\zeta^{\frac{1}{2}}(x_{\theta_e}, x_{\theta_{e+1}}\dots x_{\theta_{e+i-1}})\,\zeta^{\frac{1}{2}}(x_{\theta_{e+i}}\dots x_{\theta_{\tau-1}})\end{Bmatrix}.$$

The mark (?) is used to denote (–) raised to a power whose index is the number of exchanges of place whereby the arrangement 1, 2 ... $(\tau-1)$ can be shifted into the arrangement $\theta_1, \theta_2 \dots \theta_{\tau-1}$.

In like manner, let

$$K_2 = \Sigma(?)\begin{Bmatrix} l_{\theta_1}\dots l_{\theta_e};\ m_{\theta_{e+1}}\dots m_{\theta_{e+i-1}};\ n_{\theta_{e+i}}\dots n_{\theta_{\tau-1}} \\ \zeta^{\frac{1}{2}}(x_{\theta_1}, x_{\theta_2}\dots x_{\theta_e})\,\zeta^{\frac{1}{2}}(x, x_{\theta_{e+1}}\dots x_{\theta_{e+i-1}})\,\zeta^{\frac{1}{2}}(x_{\theta_{e+i}}\dots x_{\theta_{\tau-1}})\end{Bmatrix},$$

and $$K_3 = \Sigma(?)\begin{Bmatrix} l_{\theta_1}\dots l_{\theta_e};\ m_{\theta_{e+1}}\dots m_{\theta_{e+i}};\ n_{\theta_{e+i+1}}\dots n_{\theta_{\tau-1}} \\ \zeta^{\frac{1}{2}}(x_{\theta_1}, x_{\theta_2}\dots x_{\theta_e})\,\zeta^{\frac{1}{2}}(x_{\theta_{e+1}}, x_{\theta_{e+2}}\dots x_{\theta_{e+i}})\,\zeta^{\frac{1}{2}}(x, x_{\theta_{e+i+1}}\dots x_{\theta_{\tau-1}})\end{Bmatrix}.$$

Then, using c to denote any arbitrary constant, we shall have

$$U = cK_1,\quad V = (-)^e cK_2,\quad W = (-)^{e+i} cK_3;$$

and so, in general, the ratios to one another of any number of functions of one variable, of which the linear conjunctives for a sufficient number of given values of the variable and of the coefficients of conjunction are known to vanish, may be expressed in terms of those values.