47.

ON A PROBLEM IN TACTIC WHICH SERVES TO DISCLOSE THE EXISTENCE OF A FOUR-VALUED FUNCTION OF THREE SETS OF THREE LETTERS EACH.

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At page 375 of the May Number of this *Magazine** (in that paragraph commencing at the middle of the page) I gave a Table of Synthemes, correct as far as it went, but left in a very imperfect state. It was intended to be supplemented with a material addition which escaped my recollection when, after a long delay, the proofs of the paper passed through my hands. The question to which this Table refers is the following:—

Three *nomes*, each containing three elements, are given; the number of *trinomial* triads (that is, ternary combinations, composed by taking one element out of each nome) will be 27, and these 27 may be *grouped* together into 9 synthemes (each syntheme consisting of 3 of the triads in question, which together include between them all the 9 elements). It is desirable to know:—1st. How many distinct *groupings* of this kind can be formed. 2nd. Whether there is more than one, and, if so, how many distinct types of groupings. The criterion of one grouping being cotypal or allotypal to another is its capability or incapability of being transformed into that other by means of an interchange of elements. Be it once for all stated that the question in hand is throughout one of combinations, and not of permutations; the order of the elements in a triad, of a triad in a syntheme, of a syntheme in a grouping is treated as immaterial. As we are only concerned with the elements as distributed into *nomes*, the number of interchanges of elements with which we are concerned is 6×6^3 or 1296; the factor 6^3 arises from the permutability of the elements of each nome *inter se*, the remaining factor 6 from the permutability of any nome with any other. I find, by a method which carries its own demonstration with it on its face, that the number of distinct *groupings* is 40, of which 4 belong to one *type* or *family*, and 36 to a second type or family.

[* p. 270 above.]

Let the nomes be 1 . 2 . 3, 4 . 5 . 6, 7 . 8 . 9, and let

$$\begin{array}{llll}
c_1 \text{ denote } 1.4,\ 2.5,\ 3.6 & & \dot{c}_1 \text{ denote } 1.4,\ 2.6,\ 3.5 \\
c_2 \quad ,, \quad 1.5,\ 2.6,\ 3.4 & & \dot{c}_2 \quad ,, \quad 1.5,\ 2.4,\ 3.6 \\
c_3 \quad ,, \quad 1.6,\ 2.4,\ 3.5 & & \dot{c}_3 \quad ,, \quad 1.6,\ 2.5,\ 3.4
\end{array}$$

$$\gamma \text{ denote } \begin{matrix} 7, 8, 9 \\ 8, 9, 7 \\ 9, 7, 8 \end{matrix} \qquad \gamma' \text{ denote } \begin{matrix} 7, 9, 8 \\ 9, 8, 7 \\ 8, 7, 9 \end{matrix}$$

$$\begin{array}{llll}
b_1 \text{ denote } 1.7,\ 2.8,\ 3.9 & & \dot{b}_1 \text{ denote } 1.7,\ 2.9,\ 3.8 \\
b_2 \quad ,, \quad 1.8,\ 2.9,\ 3.7 & & \dot{b}_2 \quad ,, \quad 1.8,\ 2.7,\ 3.9 \\
b_3 \quad ,, \quad 1.9,\ 2.7,\ 3.8 & & \dot{b}_3 \quad ,, \quad 1.9,\ 2.8,\ 3.7
\end{array}$$

$$\beta \text{ denote } \begin{matrix} 4, 5, 6 \\ 5, 6, 4 \\ 6, 4, 5 \end{matrix} \qquad \beta' \text{ denote } \begin{matrix} 4, 6, 5 \\ 6, 5, 4 \\ 5, 4, 6 \end{matrix}$$

$$\begin{array}{llll}
a_1 \text{ denote } 4.7,\ 5.8,\ 6.9 & & \dot{a}_1 \text{ denote } 4.7,\ 5.9,\ 6.8 \\
a_2 \quad ,, \quad 4.8,\ 5.9,\ 6.7 & & \dot{a}_2 \quad ,, \quad 4.8,\ 5.7,\ 6.9 \\
a_3 \quad ,, \quad 4.9,\ 5.7,\ 6.8 & & \dot{a}_3 \quad ,, \quad 4.9,\ 5.8,\ 6.7
\end{array}$$

$$\alpha \text{ denote } \begin{matrix} 1, 2, 3 \\ 2, 3, 1 \\ 3, 1, 2 \end{matrix} \qquad \alpha' \text{ denote } \begin{matrix} 1, 3, 2 \\ 2, 3, 1 \\ 3, 1, 2. \end{matrix}$$

I take first the larger family of 36 groupings; these may be represented as follows:—

$a_1\alpha$	$a_1\alpha$	$a_1\alpha'$	$a_1\alpha$	$a_1\alpha'$	$a_1\alpha'$	$\dot{a}_1\alpha$	$\dot{a}_1\alpha$	$\dot{a}_1\alpha'$	$\dot{a}_1\alpha$	$\dot{a}_1\alpha'$	$\dot{a}_1\alpha'$
$a_2\alpha$	$a_2\alpha'$	$a_2\alpha$	$a_2\alpha'$	$a_2\alpha$	$a_2\alpha'$	$\dot{a}_2\alpha$	$\dot{a}_2\alpha'$	$\dot{a}_2\alpha$	$\dot{a}_2\alpha'$	$\dot{a}_2\alpha$	$\dot{a}_2\alpha'$
$a_3\alpha'$	$a_3\alpha$	$a_3\alpha$	$a_3\alpha'$	$a_3\alpha'$	$a_3\alpha$	$\dot{a}_3\alpha'$	$\dot{a}_3\alpha$	$\dot{a}_3\alpha$	$\dot{a}_3\alpha'$	$\dot{a}_3\alpha'$	$\dot{a}_3\alpha$
$b_1\beta$	$b_1\beta$	$b_1\beta'$	$b_1\beta$	$b_1\beta'$	$b_1\beta'$	$\dot{b}_1\beta$	$\dot{b}_1\beta$	$\dot{b}_1\beta'$	$\dot{b}_1\beta$	$\dot{b}_1\beta'$	$\dot{b}_1\beta'$
$b_2\beta$	$b_2\beta'$	$b_2\beta$	$b_2\beta'$	$b_2\beta$	$b_2\beta'$	$\dot{b}_2\beta$	$\dot{b}_2\beta'$	$\dot{b}_2\beta$	$\dot{b}_2\beta'$	$\dot{b}_2\beta$	$\dot{b}_2\beta'$
$b_3\beta'$	$b_3\beta$	$b_3\beta$	$b_3\beta'$	$b_3\beta'$	$b_3\beta$	$\dot{b}_3\beta'$	$\dot{b}_3\beta$	$\dot{b}_3\beta$	$\dot{b}_3\beta'$	$\dot{b}_3\beta'$	$\dot{b}_3\beta$
$c_1\gamma$	$c_1\gamma$	$c_1\gamma'$	$c_1\gamma$	$c_1\gamma'$	$c_1\gamma'$	$\dot{c}_1\gamma$	$\dot{c}_1\gamma$	$\dot{c}_1\gamma'$	$\dot{c}_1\gamma$	$\dot{c}_1\gamma'$	$\dot{c}_1\gamma'$
$c_2\gamma$	$c_2\gamma'$	$c_2\gamma$	$c_2\gamma'$	$c_2\gamma$	$c_2\gamma'$	$\dot{c}_2\gamma$	$\dot{c}_2\gamma'$	$\dot{c}_2\gamma$	$\dot{c}_2\gamma'$	$\dot{c}_2\gamma$	$\dot{c}_2\gamma'$
$c_3\gamma'$	$c_3\gamma$	$c_3\gamma$	$c_3\gamma'$	$c_3\gamma'$	$c_3\gamma$	$\dot{c}_3\gamma'$	$\dot{c}_3\gamma$	$\dot{c}_3\gamma$	$\dot{c}_3\gamma'$	$\dot{c}_3\gamma'$	$\dot{c}_3\gamma$.

An example of the development of any one of the above symbolisms into its correspondent grouping will serve to render perfectly intelligible the whole Table.

Let it be required to develope

$$\begin{matrix} \dot{b}_1\beta \\ \dot{b}_2\beta' \\ \dot{b}_3\beta'. \end{matrix}$$

Since

$$\begin{array}{llll} \dot b_1 = 1.7 \;\; 2.9 \;\; 3.8 & & 4,\ 5,\ 6 & & 4,\ 6,\ 5 \\ \dot b_2 = 1.8 \;\; 2.7 \;\; 3.9 & \beta = & 5,\ 6,\ 4 & \beta' = & 6,\ 5,\ 4 \\ \dot b_3 = 1.9 \;\; 2.8 \;\; 3.7 & & 6,\ 4,\ 5 & & 5,\ 4,\ 6, \end{array}$$

the development required is the following:—

$$\left|\begin{array}{lll} 1.7.4 & 2.9.5 & 3.8.6 \\ 1.7.5 & 2.9.6 & 3.8.4 \\ 1.7.6 & 2.9.4 & 3.8.5 \\ 1.8.4 & 2.7.6 & 3.9.5 \\ 1.8.6 & 2.7.5 & 3.9.4 \\ 1.8.5 & 2.7.4 & 3.9.6 \\ 1.9.4 & 2.8.6 & 3.7.5 \\ 1.9.6 & 2.8.5 & 3.7.4 \\ 1.9.5 & 2.8.4 & 3.7.6 \end{array}\right|.$$

The whole of this family of 36 may be represented under the following condensed form, according to the notation usual in the theory of substitutions.

$$\left(\left|\begin{array}{c} a_1\alpha \\ a_2\alpha \\ a_3\alpha \end{array}\right| \times \begin{pmatrix} 123 & 123 & 123 \\ 123 & 231 & 312 \end{pmatrix} \times \begin{pmatrix} a & \dot a \\ a & a \end{pmatrix} \times \begin{pmatrix} \alpha\alpha' & \alpha\alpha' \\ \alpha\alpha' & \alpha'\alpha \end{pmatrix} \times \begin{pmatrix} a\alpha & b\beta & c\gamma \\ a\alpha & a\alpha & a\alpha \end{pmatrix}\right).$$

It remains to describe the principal and most symmetrical family. This contains only 4 groupings, and may be represented indifferently under any of the three following forms:—

$$\begin{array}{llll} a_1\alpha\ a_1\alpha'\ \dot a_1\alpha\ \dot a_1\alpha' & & b_1\beta\ b_1\beta'\ \dot b_1\beta\ \dot b_1\beta' & & c_1\gamma\ c_1\gamma'\ \dot c_1\gamma\ \dot c_1\gamma' \\ a_2\alpha\ a_2\alpha'\ \dot a_2\alpha\ \dot a_2\alpha' & \text{or} & b_2\beta\ b_2\beta'\ \dot b_2\beta\ \dot b_2\beta' & \text{or} & c_2\gamma\ c_2\gamma'\ \dot c_2\gamma\ \dot c_2\gamma' \\ a_3\alpha\ a_3\alpha'\ \dot a_3\alpha\ \dot a_3\alpha' & & b_3\beta\ b_3\beta'\ \dot b_3\beta\ \dot b_3\beta' & & c_3\gamma\ c_3\gamma'\ \dot c_3\gamma\ \dot c_3\gamma'. \end{array}$$

In developing, it will be found that each of these three representations gives rise to the *same* family of groupings, which from its importance it is proper to set out in full as follows:—

1.4.7 2.5.8 3.6.9	1.4.7 2.5.9 3.6.8	1.4.7 2.6.8 3.5.9	1.4.7 2.6.9 3.5.8
1.4.8 2.5.9 3.6.7	1.4.9 2.5.8 3.6.7	1.4.8 2.6.9 3.5.7	1.4.9 2.6.8 3.5.7
1.4.9 2.5.7 3.6.8	1.4.8 2.5.7 3.6.9	1.4.9 2.6.7 3.5.8	1.4.8 2.6.7 3.5.9
1.5.7 2.6.8 3.4.9	1.5.7 2.6.9 3.4.8	1.5.7 2.4.8 3.6.9	1.5.7 2.4.9 3.6.8
1.5.8 2.6.9 3.4.7	1.5.9 2.6.8 3.4.7	1.5.8 2.4.9 3.6.7	1.5.9 2.4.8 3.6.7
1.5.9 2.6.7 3.4.8	1.5.8 2.6.7 3.4.9	1.5.9 2.4.7 3.6.8	1.5.8 2.4.7 3.6.9
1.6.7 2.4.8 3.5.9	1.6.7 2.4.9 3.5.8	1.6.7 2.5.8 3.4.9	1.6.7 2.5.9 3.4.8
1.6.8 2.4.9 3.5.7	1.6.9 2.4.8 3.5.7	1.6.8 2.5.9 3.4.7	1.6.9 2.5.8 3.4.7
1.6.9 2.4.7 3.5.8	1.6.8 2.4.7 3.5.9	1.6.9 2.5.7 3.4.8	1.6.8 2.5.7 3.4.9

It follows at once from the above Table, that if 3 cubic equations be given, we may form a function of the 9 roots, which, when any of the roots of any of the equations are interchanged *inter se*, or all the roots of one with all those of any other, will receive only *four distinct values*.

It also follows that we may form with 9 letters an intransitive group (of Cauchy) containing $\frac{216}{4}$, that is, 54, or a transitive group containing $\frac{1296}{4}$, or 324 substitutions. So the family of 36 groupings lead to the formation of an intransitive substitution group of $\frac{216}{12}$, that is, 18, and of a transitive group of $\frac{1296}{36}$, or 36 substitutions.

Since 9 letters may be thrown, in $\frac{8 . 7}{2} \times \frac{5 . 4}{2}$, that is, 280 different ways, into nomes of 3 letters each, it further follows that by repeating each of the above two families 280 times we shall obtain new families remaining unaltered by any substitution of any of the nine elements *inter se*, and consequently indicating the existence of substitution-groups containing

$$\frac{1 . 2 . 3 . 4 . 5 . 6 . 7 . 8 . 9}{280 \times 36} \text{ and } \frac{1 . 2 . 3 . 4 . 5 . 6 . 7 . 8 . 9}{280 \times 4},$$

that is, 36 and 324 substitutions respectively.

In the above solution a little consideration will show that the method is essentially based on the solution of a *previous* question, namely, of grouping together the synthemes of *binomial duads* of *two* nomes of three letters each, which can be done in two distinct modes, which (if, for example, we take 1 . 2 . 3, 4 . 5 . 6 as the two nomes in question) are represented in the notation used above by $\begin{matrix} c_1 \\ c_2 \\ c_3 \end{matrix}$ and $\begin{matrix} \dot{c}_1 \\ \dot{c}_2 \\ \dot{c}_3 \end{matrix}$ respectively. So, more generally, the groupings of the q-nomial q-ads of r nomes of s elements may be made to depend on the groupings of the $(q-1)$-nomial $(q-1)$-ads of $(r-1)$ nomes of s elements each. The more general question is to discover the groupings and their families of the synthemes composed of p-nomial q-ads of r nomes of s elements, of which the simplest example next that which has been considered and solved is to discover the groupings of the synthemes composed of 54 *binomial* triads of 3 nomes of 3 elements each*.

The chief difficulty of calculating *à priori* the number of such groupings is of a similar nature to that which lies at the bottom of the ordinary theory of the partition of numbers, namely, the liability of the same groupings to make their appearance under distinct symbolical representations. Of this we have seen an example in the threefold representation of the principal family

* I have ascertained, by a direct analytical method, since the above was written, that the number of different groupings of the synthemes composed of these binomial triads is 144. The number of distinct types or families is *three*, one containing 12, another 24, and the third 108 groupings.

of 4 groupings just treated of. But for the existence of this multiform representation of the same grouping we could have affirmed *à priori* the number of groupings to be $2 \times 3 \times 2^3$ or 48, whereas the true number is only 40. I believe that the above is the first instance of the doctrine of types making its appearance explicitly, and illustrated by example in the theory of tactic. It were much to be desired that some one would endeavour to collect and collate the various solutions that have been given of the noted 15-school-girl problem by Messrs Kirkman (in the *Ladies' Diary*), Moses Ansted (in the *Cambridge and Dublin Mathematical Journal*), by Messrs Cayley and Spottiswoode (in the *Philosophical Magazine* and elsewhere), and Professor Pierce, the latest and probably the best (in the *American Astronomical Journal*), besides various others originating and still floating about in the fashionable world (one, if not two, of which I remember having been communicated to me many years ago by Mr Archibald Smith, F.R.S.), with a view to ascertaining whether they belong to the same or to distinct types of aggregation.