66.

NOTE ON A THEOREM OF THE INTEGRAL CALCULUS.

[*Philosophical Magazine*, XXVI. (1863), pp. 293, 294.]

I PROPOSE briefly to lay before the mathematical readers of the Magazine a wide generalization, and at the same time a more precise statement, of the theorem contained at the close of my paper in the last Number. The theorem, as therein enunciated, was drawn from geometrical considerations, it having first manifested itself dimly to the author by a sort of indirect reflection from a metrical theory of perspective. I have since obtained a very easy proof of it in its extended form, which in spirit amounts to a free algebraical paraphrase of the method indicated in the final footnote of the paper in question. The ultimate form of the perfected theorem is particularly interesting from its simplicity of application, and from its connexion with the grand and growing theory of invariants. The proof of it will appear in its proper place in the continuation of the paper in which, in its incipient state, it first came to light*.

Theorem.—Let a figure, whether plane, solid, or hyperspatial, be supposed to be limited by a locus or loci defined by one or more algebraical equations, not necessarily the most general of their respective degrees, but each at least the most general of its degree and kind†, and let the density at any point of the figure be any *homogeneous* function of the coordinates, and let the mass of such figure be supposed to be known in terms of the constants which enter into the defining equations; next let the density at each point of the mass be multiplied by a new factor, which may be any rational integral homogeneous function of the coordinates. Then the theorem affirms that the expression

* Strange cradle this for the inception of a quasi-invariantive theory of integration, "A geometrical construction of the centre of gravity of a truncated pyramid"! Où la vérité va-t-elle se nicher?

† By *kind* I mean descriptive character, that is such character as is not affected by perspective or homographical deformation. Thus, for example, the case of a cone may be treated apart from the more general case of a surface of the second degree. So, again, a curve of the third degree with a multiple point, or having one or both of its fundamental invariants zero, may be treated apart from the case of a general cubic curve.

for the new mass may be obtained by operating upon the expression for the original one with differential operators precisely identical with combinations of certain of those which serve to define an invariant of the given system of equations, and which will be found set forth in my paper "On the Calculus of Forms," in* the *Cambridge and Dublin Mathematical Journal*†. Thus, for example, by means of the known expressions for the area or content of a triangle, ellipse, pyramid, ellipsoid, or cone, this theorem enables us by differentiation and algebraical processes alone to obtain the parameters which define the centres of gravity, moments of inertia, principal axes, &c., of such figures.

I must add an important observation, namely, that the theorem remains true when one of the defining equations (supposing there to be more than one), instead of being the most general of a certain degree and kind, is affected with arbitrary numerical coefficients (zeros or others), provided only that it be *homogeneous* in the variables. Again, the theorem continues to hold when the original density, instead of being a homogeneous function of the variables, is such function multiplied by any Covariant of the defining equations taken separately or in groups—using the word covariant in its most extended sense, so as to comprehend fractional and irrational as well as integral forms,—the only effect of the introduction of such new factor into the density being to modify the form of the differential operators. There are certain very special cases, to which it is not necessary to allude here in detail, in which the theorem becomes illusory: such will be the case, for example, for a plane area when the given density is a homogeneous function in the variables of the negative degree 3, and for a solid content when that density is of the negative degree 4‡.

* [Volume I. of this Reprint, p. 356.]

† The partial differential equations for invariants, covariants, and contravariants will be found therein stated with absolute generality for any number of functions and any number of variables. Dr Aronhold, in the last Number of *Crelle's Journal*, states erroneously that these equations were given by me for binary functions only, and subsequently generalized by Cayley and Clebsch.

‡ A similar method applied to *extents* (as curves, surfaces, &c.) gives rise to curious theorems. Thus I find that the mass of a plane curve affected with a density varying at each point as the square of the cosine of the inclination of the curve to a fixed line, is a differential derivative of the length of the curve. So, again, the moment of inertia of a curve in respect to any axis perpendicular to its plane, is a differential derivative of its moment in respect to an arbitrary line in its plane.