ERRATUM

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1) Last term in (3.4) should read

$$-\frac{1}{3K}\delta_{ij}\dot{U} \quad \text{instead of} \quad +\frac{1}{3K}\delta_{ij}\dot{U};$$

2) Eq. (3.7) (stress functions in the Hencky-Ilyushin theory of plasticity) should read

$$3(\varphi_{,\alpha\beta}+\varphi_{,\alpha}\partial_\beta+\varphi_{,\beta}\partial_\alpha+\varphi\partial^2_{\alpha\beta})(\phi_{ij,\alpha\beta}+\phi_{\alpha\beta,ij}-\phi_{i\alpha,j\beta}-\phi_{j\beta,i\alpha})$$
$$+\left[\varphi_{,ij}-\delta_{ij}(\nabla^2\varphi+2\varphi_{,\mu}\partial_\mu)+\varphi_{,i}\partial_j+\varphi_{,j}\partial_i+\left(\varphi-\frac{1}{3K}\right)(\partial^2_{ij}-\delta_{ij}\nabla^2)\right]$$
$$\times(\nabla^2\phi_{\alpha\alpha}-\phi_{\alpha\beta,\alpha\beta})+\frac{1}{K}(U_{,ij}-\delta_{ij}\nabla^2 U)=0.$$

Authors express regret for these errors.

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Basic equations of perfect plasticity parametrized by general stress functions

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THE PAPER derives governing equations of perfect plasticity for the general Finzi-Blokh-Krutkov stress functions (1.1), parametrizing equations of internal equilibrium. The Huber-Mises-Hencky yield condition takes the form (2.7) irrespectively of the assumed theory of plasticity; compatibility equations turn into Eq. (3.6) for the Prandtl-Reuss theory, and into Eq. (3.7) for the Hencky-Ilyushin theory of plasticity. The system obtained consists of four independent equations for four unknows: three stress functions (chosen out of six), and the function λ or φ.

1. General remarks

FUNCTIONS parametrizing the equations of internal equilibrium are called stress functions. Since equilibrium equations are identical in all branches of mechanics of solids, stress functions are also introduced in an identical manner; the differences are seen only in final governing equations.

The general approach to stress functions was worked out by B. FINZI [2], V. I. BLOKH [1] and Yu. A. KRUTKOV [4]; they proved that in the case of potential body forces the following tensor of stress functions ϕ_{ij} satisfies identically the equilibrium equations

$$\sigma_{ij} = e_{i\alpha\mu} e_{j\beta\nu} \phi_{\alpha\beta,\mu\nu} - \delta_{ij} U. \tag{1.1}$$

In Eq. (1.1) $e_{i\alpha\mu}$ and $e_{j\beta\nu}$ are the permutation symbols (alternators), δ_{ij} denotes the Kronecker symbol, U — potential of body forces. Moreover, the summation convention is employed. Notation (1.1) is valid for Cartesian coordinates and only this case will be dealt with here; the generalization of Eq. (1.1) to arbitrary curvilinear coordinates, also given by B. Finzi, is not difficult but final equations are much longer and more complicated.

The system of stress functions (1.1) is complete for a simply connected body. In general, if the boundary consists of several closed surfaces, this system is not complete. A suitable generalization was given by N. E. GURTIN [3]. In our case we confine our considerations to the functions ϕ_{ij}.

Yu. A. KRUTKOV [4] discussed in detail the basic equations of the theory of elasticity expressed in terms of the stress functions ϕ_{ij}. In the theory of plasticity some particular cases of Eq. (1.1), namely the Airy and the Prandtl stress functions, have been used frequently, but the general case has not been considered up till now. The derivation of governing equations of perfect plasticity for general stress functions is the aim of the present paper and all the classical theories (Prandtl-Reuss, Levy-Mises, Hencky-Ilyushin) will be considered.

2. The Huber-Mises-Hencky yield condition

The HMH yield condition will be used in the form

$$s_{\alpha\beta}s_{\alpha\beta} = \frac{2}{3}\sigma_0^2 \tag{2.1}$$

where

$$s_{ij} = \sigma_{ij} - \delta_{ij}\sigma_m, \tag{2.2}$$

σ_m denotes the mean stress, and σ_0 — the yield-point stress. To introduce Eq. (1.1) into Eq. (2.1) we first determine the mean stress

$$\sigma_m = \frac{1}{3}\sigma_{\gamma\gamma} = \frac{1}{3}e_{\gamma\alpha\mu}e_{\gamma\beta\nu}\phi_{\alpha\beta,\mu\nu} - U. \tag{2.3}$$

Making use of the identity

$$e_{\gamma\alpha\mu}e_{\gamma\beta\nu} = \delta_{\alpha\beta}\delta_{\mu\nu} - \delta_{\alpha\nu}\delta_{\beta\mu} \tag{2.4}$$

we obtain finally

$$\sigma_m = \frac{1}{3}(\phi_{\alpha\alpha,\beta\beta} - \phi_{\alpha\beta,\alpha\beta}) - U. \tag{2.5}$$

Substituting Eqs. (1.1) and (2.5) into (2.1) we may now express the deviatoric stresses

$$s_{ij} = e_{i\alpha\mu}e_{j\beta\nu}\phi_{\alpha\beta,\mu\nu} - \frac{1}{3}\delta_{ij}(\phi_{\alpha\alpha,\beta\beta} - \phi_{\alpha\beta,\alpha\beta}). \tag{2.6}$$

Further, substituting Eq. (2.6) into the HMH yield condition (2.1), we obtain after contraction and some rearrangements its final form

$$3\phi_{\alpha\beta,\mu\nu}(\phi_{\alpha\beta,\mu\nu} + \phi_{\mu\nu,\alpha\beta} - \phi_{\alpha\mu,\beta\nu} - \phi_{\beta\nu,\alpha\mu}) - (\phi_{\alpha\alpha,\beta\beta} - \phi_{\alpha\beta,\alpha\beta})^2 = 2\sigma_0^2. \tag{2.7}$$

If the problem under consideration is internally statically-determinate, then Eq. (2.7) is sufficient for solution. For example, for the Airy stress function $\phi_{zz} = \phi_{zz}(x, y)$ putting in Eq. (2.7) other stress functions equal to zero, we obtain one equation with one unknown

$$\left(\frac{\partial^2\phi_{zz}}{\partial x^2}\right)^2 + \left(\frac{\partial^2\phi_{zz}}{\partial y^2}\right)^2 - \frac{\partial^2\phi_{zz}}{\partial x^2}\frac{\partial^2\phi_{zz}}{\partial y^2} + 3\left(\frac{\partial^2\phi_{zz}}{\partial x\partial y}\right)^2 = \sigma_0^2. \tag{2.8}$$

Retaining in Eq. (2.7) only two stress functions

$$\phi_{xz} = \phi_{xz}(x, y) \quad \text{and} \quad \phi_{yz} = \phi_{yz}(x, y), \tag{2.9}$$

we obtain

$$3\left(\frac{\partial^2\phi_{yz}}{\partial x^2} - \frac{\partial^2\phi_{xz}}{\partial x\partial y}\right)^2 + 3\left(\frac{\partial^2\phi_{yz}}{\partial x\partial y} - \frac{\partial^2\phi_{xz}}{\partial y^2}\right)^2 = \sigma_0^2$$

and hence for the Prandtl function Ψ

$$\Psi = \frac{\partial\phi_{yz}}{\partial x} - \frac{\partial\phi_{xz}}{\partial y}, \tag{2.10}$$

we derive the well-known equation of the surface of constant slope

$$3\left(\frac{\partial\Psi}{\partial x}\right)^2 + 3\left(\frac{\partial\Psi}{\partial y}\right)^2 = \sigma_0^2. \tag{2.11}$$

In the case of an internally statically-indeterminate problem we have to take the relations between strains and stresses into account. Using these relations and substituting the resulting expressions for strains into the Cauchy formulae we obtain a system with 7 unknows, whereas the substitution into the compatibility equations leads to a system with only 4 unknows. Here we follow the latter way, though in certain cases it is impossible to evaluate all integration constants without effectively calculating displacements or velocities.

3. Compatibility equations

First we apply the Prandtl-Reuss equations as the most complicated law. Making use of the law of volume change

$$\dot{\varepsilon}_m = \frac{1}{3K}\dot{\sigma}_m \tag{3.1}$$

and substituting it into the Prandtl-Reuss equations

$$\dot{\varepsilon}_{ij} - \delta_{ij}\dot{\varepsilon}_m = \lambda s_{ij} + \frac{1}{2G}\dot{s}_{ij} \tag{3.2}$$

we present them in the form

$$\dot{\varepsilon}_{ij} = \lambda(\sigma_{ij} - \delta_{ij}\sigma_m) + \frac{1}{2G}\dot{\sigma}_{ij} - \left(\frac{1}{2G} - \frac{1}{3K}\right)\delta_{ij}\dot{\sigma}_m. \tag{3.3}$$

Substituting here Eqs. (1.1) and (2.5), we may write

$$\dot{\varepsilon}_{ij} = \lambda\left[e_{i\alpha\mu}e_{j\beta\nu}\phi_{\alpha\beta,\mu\nu} - \frac{1}{3}\delta_{ij}(\phi_{\alpha\alpha,\beta\beta} - \phi_{\alpha\beta,\alpha\beta})\right] + \frac{1}{2G}e_{i\alpha\mu}e_{j\beta\nu}\dot{\phi}_{\alpha\beta,\mu\nu} - \frac{\nu}{E}\delta_{ij}(\dot{\phi}_{\alpha\alpha,\beta\beta} - \dot{\phi}_{\alpha\beta,\alpha\beta}) + \frac{1}{3K}\delta_{ij}\dot{U}, \tag{3.4}$$

where ν as a multiplier denotes Poisson's ratio and is not to be confused with ν as a dummy index.

Now we will substitute Eq. (3.4) into the compatibility equations in the form

$$\dot{\varepsilon}_{\alpha\beta,\mu\nu}e_{i\alpha\mu}e_{j\beta\nu} = 0, \tag{3.5}$$

which has only two free indices and gives only 9 equations, 6 of which are distinct (instead of four free indices as in usual notation). We obtain the following system of equations for the Prandtl-Reuss theory:

$$3\left[\lambda_{,\alpha\beta} + \lambda_{,\alpha}\partial_\beta + \lambda_{,\beta}\partial_\alpha + \left(\lambda + \frac{1}{2G}\frac{d}{dt}\right)\partial^2_{\alpha\beta}\right](\phi_{ij,\alpha\beta} + \phi_{\alpha\beta,ij} - \phi_{i\alpha,j\beta} - \phi_{j\beta,i\alpha}) + \left[\lambda_{,ij} - \delta_{ij}(\nabla^2\lambda + 2\lambda_{,\mu}\partial_\mu) + \lambda_{,i}\partial_j + \lambda_{,j}\partial_i + \left(\lambda + \frac{3\nu}{E}\frac{d}{dt}\right)(\partial^2_{ij} - \delta_{ij}\nabla^2)\right]\cdot(\nabla^2\phi_{\alpha\alpha} - \phi_{\alpha\beta,\alpha\beta}) + \frac{1}{K}(\dot{U}_{,ij} - \delta_{ij}\nabla^2\dot{U}) = 0, \tag{3.6}$$

where the operator ∂_α denotes partial differentiation with respect to the coordinate α (or x_α).

The corresponding equations for the Levy-Mises theory may be derived directly from Eq. (3.6) if we substitute $\frac{1}{G} = \frac{1}{E} = \frac{1}{K} = 0$. For the Hencky-Ilyushin theory one obtains

$$(3.7) \quad 3\varphi(\phi_{ij,\alpha\alpha\beta\beta}+\phi_{\alpha\beta,ij\alpha\beta}-\phi_{i\alpha,j\alpha\beta\beta}-\phi_{j\beta,i\beta\alpha\alpha}) + \left(\varphi-\frac{1}{3K}\right)(\phi_{\alpha\alpha,ij\beta\beta}-\phi_{\alpha\beta,ij\alpha\beta}-\delta_{ij}\phi_{\alpha\alpha,\beta\beta\mu\mu}+\delta_{ij}\phi_{\alpha\beta,\alpha\beta\mu\mu}) + \frac{1}{K}(U_{,ij}-\delta_{ij}U_{,\alpha\alpha}) = 0.$$

In fact only three compatibility equations are really independent (cf. L. E. MALVERN [5]) and hence the system of Eq. (2.7) with Eq. (3.6) or with Eq. (3.7) contains four independent equations. The number of unknows also amounts to four: three stress functions (chosen out of six), e.g. the Maxwell or the Morera stress functions, and the function λ or φ.

In their general form those equations are difficult to apply but in some particular cases they have already found application, for example in the problems of thick-walled tubes under combined loadings (J. SKRZYPEK [6] and M. ŻYCZKOWSKI [7]).

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