Archives of Mechanics • Archiwum Mechaniki Stosowanej, 29, 1, pp. 53-67, Warszawa 1977

Two representations of sensitivity and error analysis of a dynamic system

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A GENERAL dynamic system with a "white noise" stochastic input as well as with a variable system matrix is analysed. The variations have a deterministic or stochastic character. Two methods of solution of both problems are considered. The first one is based on the Taylor expansion given by VETTER [7]; the second one is based on the Taylor expansion given by GAWROŃSKI [3] and the new definition of a correlation matrix [3].

Przeanalizowano ogólny układ dynamiczny z wejściem stochastycznym typu «biały szum», jak również z zaburzeniami macierzy układu. Zaburzenia te mają charakter zdeterminowany lub stochastyczny. Rozważa się dwa sposoby rozwiązania zagadnienia. Pierwszy oparty jest na macierzowym rozwinięciu Taylora wg VETTERA [7]; drugi bazuje na macierzowym rozwinięciu Taylora wg GAWROŃSKIEGO [3] i nowej definicji macierzy korelacji [3].

Анализируется общая динамическая система со стохастическим входом типа ,,белый шум'', как тоже с возмущениями матрицы системы. Эти возмущения имеют детерминированный или стохастических характер. Обсуждаются два способа решения задачи. Первый опирается на матричное разложение Тейлора по ВЕТТЕРУ [7]; второй базирует на матричном разложении Тейлора по ГАВРОНСКОМУ [3] и на новом определении матрицы корреляции [3].

1. Introduction

THE PAPER presents the analysis of a dynamic system described by the equation:

$$\dot{x}(t) = A(t)x(t)+B(t)u(t), \tag{1.1}$$

with the initial value $x(t_0)$, where $x(t)$ is the state vector of order n and, at the same time, the output vector of the system, $u(t)$ is the input vector of the system of order m, $A(t)$ and $B(t)$ are the matrices $n\times n$ and $n\times m$, respectively.

The output of the system $x(t)$ is determined when the input $u(t)$ is a white noise stochastic process. On the other hand the analysis of the system with variations of the system matrix $A(t)$ is provided. These variations are deterministic or random. Their influence on the output vector $x(t)$ is analysed.

All the cases are examined in two ways. One of them is based on the Taylor expansion given by VETTER [7]; another is based on the Taylor expansion given by GAWROŃSKI [3] and a new definition of a correlation matrix [3].

2. Description of the methods

Let F and G be matrices of dimensions $p\times q$ and $s\times t$, respectively; The function

$$F = F(G) \tag{2.1}$$

is called a matrix function of a matrix argument if the entries of F depends on G:

$$f_{ij} = f_{ij}(G); \quad i = 1, \dots, p; \quad j = 1, \dots, q. \tag{2.2}$$

The first method is based on the Taylor expansion given by VETTER [7]. This expansion consists of a column transformation of the matrices F and $G^{(*)}$:

$$f = cs(F), \quad g = cs(G). \tag{2.3}$$

In this way, from the matrices F and G we obtain the column matrices f and g of order pq and st, respectively. This transformation gives the first two terms of the expansion in the form (see [7], also the Appendix)

$$f(g) = f(g_0)+T^{tg}(g-g_0), \tag{2.4}$$

where

$$T^{fg} = \left.\frac{\partial f}{\partial g^T}\right|_{g=g_0} \doteq \left[\begin{matrix} \frac{\partial f_1}{\partial g_1} & \frac{\partial f_1}{\partial g_2} & \cdots & \frac{\partial f_1}{\partial g_v} \\ \frac{\partial f_2}{\partial g_1} & \frac{\partial f_2}{\partial g_2} & \cdots & \frac{\partial f_2}{\partial g_v} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial f_u}{\partial g_1} & \frac{\partial f_u}{\partial g_2} & \cdots & \frac{\partial f_u}{\partial g_v} \end{matrix}\right]_{g=g_0} \tag{2.5}$$

and $u = pq$, $v = st$. If we denote the variations of f and g as

$$\Delta f = f-f(g_0), \quad \Delta g = g-g_0 \tag{2.6}$$

then, from Eq. (2.4) we obtain the relationship

$$\Delta f = T^{tg}\,\Delta g. \tag{2.7}$$

The matrix T^{fg} is the sensitivity matrix of the column matrix f with respect to the column matrix g, and we call it the first form sensitivity matrix.

If we use the column form f and g of matrices F and G, than, we can define their correlation matrices as

$$\begin{aligned} R_f^{(1)} &= E(\Delta f\,\Delta f^T), \\ R_g^{(1)} &= E(\Delta g\,\Delta g^T). \end{aligned} \tag{2.8}$$

These correlation matrices are called the first form correlation matrices. From the definition equations (2.8) and the relationship equation (2.4) we arrive at the following relationship between the correlation matrices $R_f^{(1)}$ and $R_g^{(1)}$:

$$R_f^{(1)} = T^{fg}R_g^{(1)}\,(T^{fg})^T. \tag{2.9}$$

The second method of the sensitivity and error analysis uses the Taylor expansion given in [3]. The first two terms of this expansion of a matrix function $F(G)$ are as follows (see the Appendix):

$$f_{ij}(G) = f_{ij}(G_0)+S_{ij}^{fG}\circ(G-G_0), \tag{2.10}$$

(*) A description of the symbols, definitions and basic relationships can be found in the Appendix.

$i = 1, \ldots, p$; $j = 1, \ldots, q$; where G_0 is a fixed value of G, f_{ij} is the ij-th entry of F, and

$$S_{ij}^{fG} = \frac{\partial f_{ij}}{\partial G}\bigg|_{G=G_0} = \begin{bmatrix} \frac{\partial f_{ij}}{\partial g_{11}} & \frac{\partial f_{ij}}{\partial g_{12}} & \cdots & \frac{\partial f_{ij}}{\partial g_{1t}} \\ \frac{\partial f_{ij}}{\partial g_{21}} & \frac{\partial f_{ij}}{\partial g_{22}} & \cdots & \frac{\partial f_{ij}}{\partial g_{2t}} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial f_{ij}}{\partial g_{s1}} & \frac{\partial f_{ij}}{\partial g_{s2}} & \cdots & \frac{\partial f_{ij}}{\partial g_{st}} \end{bmatrix}_{G=G_0} \tag{2.11}$$

If we denote

$$\Delta F = F - F(G_0), \quad \Delta G = G - G_0 \tag{2.12}$$

then, from Eq. (2.10) we obtain

$$\Delta F = [S_{ij}^{fG} \circ \Delta G], \quad i = 1, \ldots, p, \quad j = 1, \ldots, q; \tag{2.13}$$

where S_{ij}^{fG}, given by Eq. (2.11), is the sensitivity matrix of the f_{ij} entry of F with respect to the matrix G, and is called the second form sensitivity matrix.

The correlation matrices of F and G are defined as follows [3]:

$$\begin{aligned} R_F^{(2)} &= E(\Delta F \otimes \Delta F) = E(\Delta F^{\otimes 2}), \\ R_G^{(2)} &= E(\Delta G \otimes \Delta G) = E(\Delta G^{\otimes 2}). \end{aligned} \tag{2.14}$$

We call these matrices the second form correlation matrices.

From Eqs. (2.13) and (2.14) the relationship between them is

$$R_F^{(2)} = \mathscr{S}_{ijkl}^{fG} \circ R_G^{(2)}, \tag{2.15}$$

$i, k = 1, \ldots, p, j = 1, \ldots, q$, where:

$$\mathscr{S}_{ijkl}^{fG} = S_{ij}^{fG} \otimes S_{kl}^{fG}. \tag{2.16}$$

Note that the second form correlation matrix:

1. Has the same structure as its subject, i.e., correlation matrix of a scalar is a scalar, correlation matrix of a vector is a vector, and correlation matrix of a matrix is a matrix itself.

2. Is more general than the first form correlation matrix. The latter one cannot be applied to matrices unless we transform matrices into vectors.

3. Has the same definition equations for all the types. If α is a scalar, a is a vector, and A is a matrix, their second form correlation matrices are defined as follows:

$$\begin{aligned} R^{(2)} &= E\big((\alpha - \alpha_0)^{\otimes 2}\big) = E\big((\alpha - \alpha_0)^2\big) = \sigma_\alpha^2, \\ R_a^{(2)} &= E\big((a - a_0)^{\otimes 2}\big), \\ R_A^{(2)} &= E\big((A - A_0)^{\otimes 2}\big), \end{aligned}$$

where α_0, a_0, A_0 are mean values of α, a, and A. The above relationships show that the definition of the second form correlation matrices is the same for scalars, vectors and matrices. In a general case, when F is a complex function of a real argument t, its correlation matrix is defined as follows:

$$R_F^{(2)}(t_1, t_2) = E[(F(t_1)-F_0(t_1)) \otimes (\bar{F}(t_2)-\bar{F}_0(t_2))], \tag{2.17}$$

where $\bar{F}$ is the complex conjugate matrix of F, and F_0 is the mean value of F.

The correlation coefficients of the entries of the matrix F are distributed in the first and in the second form correlation matrices as follows. In the first form correlation matrix $R_f^{(1)}$ the correlation coefficient of f_{ij} and f_{kl} entries of F lies in the u-th row and v-th column of $R_f^{(1)}$, where

$$u = (j-1)p+i, \quad v = (l-1)p+k,$$

and the variance $\sigma_{f_{ij}}^2$ of the f_{ij} entry is in the u-th position in the main diagonal of $R_f^{(1)}$.

In the second form correlation matrix $R_F^{(2)}$ the correlation coefficient of f_{ij} and f_{kl} lies in the ij-th block of $R_F^{(2)}$ in the kl-th position in it. The variance of f_{ij} lies in the ij-th block of $R_F^{(2)}$ in the ij-th position in it.

3. Analysis of the dynamic system with a white noise input

The analysis of the system with a stochastic input, according to the two different definition of correlation matrices, is presented in two ways. One way is when the first form correlation matrix is applied to the solution of the problem.

Let the input $u(t)$ of the system described by Eq. (1.1) be a white noise stochastic process with a mean value

$$u_0(t) = E(u(t)) \tag{3.1a}$$

and the first form correlation matrix

$$R_u^{(1)}(t_1, t_2) = E(\Delta u(t_1) \Delta u^T(t_2)) = Q(t_1)\,\delta(t_1-t_2), \tag{3.1b}$$

where $\Delta u(t) = u(t)-u_0(t)$, and $\delta(t)$ is a Dirac delta distribution. The output mean value $x_0(t)$ and its first form correlation matrix $R_x^{(1)}(t)$ are determined if the correlation matrix of the initial value is

$$E(\Delta x(t_0) \Delta x^T(t_0)) = \Psi, \tag{3.1c}$$

where $\Delta x(t) = x(t)-x_0(t)$, and the input vector and the initial value vector are not correlated:

$$E(\Delta x(t_0) \Delta u^T(t)) = 0.$$

The mean value $x_0(t)$ is found from the solution of the equation

$$\dot{x}_0(t) = A(t)x_0(t)+B(t)u_0(t) \tag{3.2a}$$

which can be presented in the form

$$x_0(t) = \Phi(t, t_0)x_0(t_0)+\int_{t_0}^{t} \Phi(t, \tau)B(\tau)u_0(\tau)d\tau, \tag{3.2b}$$

where $\Phi(t_1, t_2)$ is the state transition matrix with the following property:

$$\Phi(t, t) = I. \tag{3.3}$$

The first form correlation matrix $R_x^{(1)}(t)$ for simplicity we denote $R_x^{(1)}(t, t)$ as $R^{(1)}(t)$ is determined from the following matrix differential equation [1, 6]:

$$\dot{R}_x^{(1)}(t) = A(t)R_x^{(1)}(t) + R_x^{(1)}(t)A^T(t) + B(t)Q(t)B^T(t) \tag{3.4}$$

with the initial condition $R_x^{(1)}(t_0) = \Psi$.

Now we solve this problem using the second form correlation matrices. The mean value of the input is given by Eq. (3.1a). The second form correlation matrix of the input vector $u(t)$ is in fact a vector itself, in the form

$$R_u^{(2)}(t_1, t_2) = E\big(\Delta u(t_1) \otimes \Delta u(t_2)\big) = q(t_1)\,\delta(t_1 - t_2). \tag{3.5a}$$

The initial value correlation matrix is

$$E\big((\Delta x(t_0))^{\otimes 2}\big) = \psi \tag{3.5b}$$

and the input vector and the initial value vector are not correlated:

$$E\big(\Delta x(t_0) \otimes \Delta u(t)\big) = 0. \tag{3.5c}$$

We obtain the mean value of $x(t)$ from Eqs. (3.2a) and (3.2b); subtracting Eq. (3.2a) from Eq. (1.1) we obtain the equation for variation of x with respect to x_0 in the form

$$\Delta\dot{x}(t) = A(t)\Delta x(t) + B(t)\Delta u(t). \tag{3.6}$$

The direct right multiplication of Eq. (3.6) by $\Delta x(t)$ gives

$$\Delta\dot{x}(t) \otimes \Delta x(t) = A(t)\Delta x(t) \otimes \Delta x(t) + B(t)\Delta u(t) \otimes \Delta x(t),$$

and from the direct left multiplication of Eq. (3.6) by $\Delta x(t)$ we acquire

$$\Delta x(t) \otimes \Delta\dot{x}(t) = \Delta x(t) \otimes A(t)\Delta x(t) + \Delta x(t) \otimes B(t)\Delta u(t).$$

From the property of the direct product (see the Appendix, eq. (A1)) we transform the latter equations into the form

$$\Delta\dot{x}(t) \otimes \Delta x(t) = \big(A(t) \otimes \underset{n}{I}\big)\big(\Delta x(t) \otimes \Delta x(t)\big) + \big(B(t) \otimes \underset{n}{I}\big)\big(\Delta u(t) \otimes \Delta x(t)\big),$$

$$\Delta x(t) \otimes \Delta\dot{x}(t) = \big(\underset{n}{I} \otimes A(t)\big)\big(\Delta x(t) \otimes \Delta x(t)\big) + \big(\underset{n}{I} \otimes B(t)\big)\big(\Delta x(t) \otimes \Delta u(t)\big).$$

The mean value operation, after adding these two equations and taking into account (A2), gives

$$\dot{R}_x^{(2)}(t) = \big(A(t) \otimes \underset{n}{I} + \underset{n}{I} \otimes A(t)\big)R_x^{(2)}(t) + \big(B(t) \otimes \underset{n}{I}\big)E\big(\Delta u(t) \otimes \Delta x(t)\big) + \\ + \big(\underset{n}{I} \otimes B(t)\big)E\big(\Delta x(t) \otimes \Delta u(t)\big). \tag{3.7}$$

The solution of Eq. (3.6) gives

$$\Delta x(t) = \Phi(t, t_0)\Delta x(t_0) + \int_{t_0}^{t} \Phi(t, \tau)B(\tau)\Delta u(\tau)\,d\tau,$$

and from (A1) we obtain

$$E\big(\Delta x(t) \otimes \Delta u(t)\big) = \big(\Phi(t, t_0) \otimes \underset{m}{I}\big)E\big(\Delta x(t) \otimes \Delta u(t)\big) \\ + \int_{t_0}^{t} \big(\Phi(t, \tau)B(\tau) \otimes \underset{m}{I}\big)E\big(\Delta u(\tau) \otimes \Delta u(t)\big)d\tau,$$

but the relationships (3.5a) and (3.5c) give

$$E(\Delta x(t) \otimes \Delta u(t)) = \int_{t_0}^{t} (\Phi(t,\tau)B(\tau) \otimes \underset{m}{I})q(t)\delta(t-\tau)d\tau = \frac{1}{2}(B(t) \otimes \underset{m}{I})q(t)$$

thus, because of Eq. (3.3) and the symmetry of $\delta(t)$. In a similar way we obtain

$$E(\Delta u(t) \otimes \Delta x(t)) = \frac{1}{2}(\underset{m}{I} \otimes B(t))q(t).$$

The last two equations as well as Eq. (3.7) give

$$\dot{R}_x^{(2)}(t) = \mathscr{A}(t)R_x^{(2)}(t) + \mathscr{B}(t)q(t), \tag{3.8}$$

where

$$\mathscr{A}(t) = A(t) \otimes \underset{n}{I} + \underset{n}{I} \otimes A(t), \tag{3.9}$$

$$\mathscr{B}(t) = (B(t))^{\otimes 2} = B(t) \otimes B(t), \tag{3.10}$$

and the initial value is $R_x^{(2)}(t_0) = \psi$.

Notice that the first form correlation matrix is obtained from the matrix equation (3.4), and the second form correlation matrix is determined from Eq. (3.8) which is an ordinary differential vector equation. The solution of Eq. (3.8) is simpler than that of Eq. (3.4).

Example 1. The system matrices are as follows (a simple vibratory system):

$$A = \begin{bmatrix} 0 & 1 \\ -100 & -2 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

the output is a vector of order 2: $x = \mathrm{col}\,(x_1, x_2)$, and the input $u(t)$ is a scalar white noise stochastic process with the zero mean value and the correlation matrix (in this case this matrix is a scalar):

$$R_u^{(2)}(t) = 10\delta(t).$$

The initial value $R_x^{(2)}(t_0) = 0$.

The second form correlation matrix $R_x^{(2)}(t)$ of the output $x(t)$ is determined from Eq. (3.8). From the relationships (3.9) and (3.10) we obtain

$$\mathscr{A} = \begin{bmatrix} 0 & 1 & 1 & 0 \\ -100 & -2 & 0 & 1 \\ -100 & 0 & -2 & 1 \\ 0 & -100 & -100 & -4 \end{bmatrix},$$

$$\mathscr{B} = \mathrm{col}\,(0, \quad 0 \quad 0, \quad 1).$$

The Laplace transformation of Eq. (3.8) gives

$$R_x^{(2)}(s) = \frac{10}{s}(sI - \mathscr{A})^{-1}\mathscr{B}$$

or

$$R_x^{(2)}(s) = \frac{10}{s(s+2)(s^2+4s+400)} \begin{bmatrix} 2 \\ s \\ s \\ s^2+2s+200 \end{bmatrix},$$

which gives

$$R_x^{(2)}(t) = \begin{bmatrix} r_{x11}(t) \\ r_{x12}(t) \\ r_{x21}(t) \\ r_{x22}(t) \end{bmatrix} = \begin{bmatrix} 0.0125\left(1-e^{-2t}(1.01+0.101\sin(19.90t+3.04143))\right) \\ 0.02525e^{-2t}\left(1+\sin\left(19.90t-\dfrac{\pi}{2}\right)\right) \\ 0.02525e^{-2t}\left(1+\sin\left(19.90t-\dfrac{\pi}{2}\right)\right) \\ 2.5\left(1-1.01e^{-2t}(1+0.099\sin(19.90t-0.3035))\right) \end{bmatrix}.$$

The graphs of the above functions $r_{x11}(t)$, $r_{x12}(t) = r_{x21}(t)$ and $r_{x22}(t)$ are presented in Fig. 1.

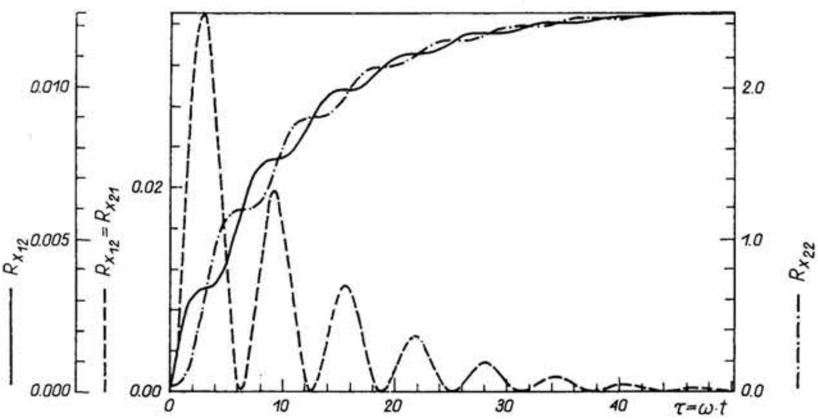

FIG. 1.

4. Analysis of the system with variations of the parameters

In the system described by the state equation (1.1) where $x(t)$ and $u(t)$ are output and input respectively, the remaining quantities, matrices $A(t)$, $B(t)$ and the initial value vector $x(t_0)$ are called parameters of the system. The parameters are assumed to be known during an investigation of the system. These parameters, however, can be subjected to variations, and these variations influence the output $x(t)$. Here, we do not examine the influence of variations of all the parameters on the output vector, but we limit ourselves to analysing the variations of the state matrix $A(t)$ only. The analysis of variations of $B(t)$ and $x(t_0)$ is similar. This assumption fixes the values of $B(t)$, $x(t_0)$ in Eq. (1.1) and in this way the vector $x(t)$ depends on the matrix $A(t)$ only:

$$x = x(A) \tag{4.1}$$

for every $t \in T$, where T is the period of system investigation.

Variations of $A(t)$ can be deterministic or random, and it is assumed, that they are small in comparison with the fixed values of parameters as a measure of magnitude of A

we use its norm. When the variations are deterministic, the analysis is called a sensitivity analysis; when the variations are random it is called an error analysis.

In the sensitivity analysis of the system we assume that a deterministic variation of $A(t)$ with respect to the fixed value $A_0(t)$ is $\Delta A(t)$. This variation results on a variation $\Delta x(t)$ of the vector $x(t)$ with respect to $x_0(t)$. The vector $x_0(t)$ is a solution of Eq. (1.1) with the fixed value $A_0(t)$.

If the deviation of parameter $A(t)$ has a random character, then, its properties can be described by its mean value and correlation matrix. An error analysis is understood here as a description of the relationships between the correlation matrix of the matrix $A(t)$ and the correlation matrix of the vector $x(t)$.

If we compare the definitions of the sensitivity matrices of the first and the second form (see Eq. (2.5) and (2.11)), we notice that their entries are those from a matrix:

$$S^{FG} = \left.\frac{\partial F}{\partial G}\right|_{G=G_0}, \tag{4.2}$$

which we call the general sensitivity matrix. Deriving the general sensitivity matrix from Eq. (4.2) and, in the second step, rearranging its entries we obtain T^{fg} and S_{ij}^{fG} matrices $(i = 1, \ldots, p, j = 1, \ldots, q)$.

For the function $x(A)$ (see Eq. (4.1)) we derive the general sensitivity matrix

$$S^{xA} = \left.\frac{\partial x}{\partial A}\right|_{A=A_0}. \tag{4.3}$$

We obtain this matrix through the differentiation of both sides of Eq. (1.1) with respect to A, for fixed t:

$$\frac{d}{dt}\left(\frac{\partial x}{\partial A}\right) = \frac{\partial A}{\partial A}(\underset{n}{I}\otimes x)+(\underset{n}{I}\otimes A)\frac{\partial x}{\partial A}+\frac{\partial B}{\partial A}(\underset{n}{I}\otimes u). \tag{4.4}$$

We have assumed that input $u(t)$ does not depend upon A. In this equation we have used the relationship (A3). If we take into account

$$\frac{\partial A}{\partial A}(\underset{n}{I}\otimes x) = E_{n\times n}^{n\times n}(\underset{n}{I}\otimes x) = x\otimes \underset{n}{I},$$

and if we examine Eq. (4.4) for $A = A_0$, we obtain the differential equation

$$\dot{S}^{xA}(t) = H(t)S^{xA}(t)+S^{BA}(t)\,U(t)+X(t) \tag{4.5}$$

with the initial value $S^{xA}(t_0)$, where $S^{xA}(t)$ is defined by Eq. (4.3), and

$$\begin{aligned} S^{BA}(t) &= \left.\frac{\partial B}{\partial A}\right|_{A=A_0}, \\ H(t) &= \underset{n}{I}\otimes A_0(t), \\ U(t) &= \underset{n}{I}\otimes u(t), \\ X(t) &= x_0(t)\otimes \underset{n}{I}, \\ S^{xA}(t_0) &= \left.\frac{\partial x(t_0)}{\partial A}\right|_{A=A_0}. \end{aligned} \tag{4.6}$$

The matrix differential equation (4.5) is equivalent to the following set of vector differential equations:

(4.7) $$\dot{s}_{ij}^{xa}(t) \doteq A_0(t)s_{ij}^{xa}(t)+s_{ij}^{Ba}(t)u(t)+x_{0i}(t)e_j$$

with the initial value $s_{ij}^{xa}(t_0)$, where

(4.8a) $$s_{ij}^{xa} = \frac{\partial x}{\partial a_{ij}}\bigg|_{A=A_0} = \mathrm{col}\left(\frac{\partial x_1}{\partial a_{ij}}, \ldots, \frac{\partial x_n}{\partial a_{ij}}\right)\bigg|_{A=A_0}$$

is the ij-th block of S^{xA}, furthermore:

(4.8b) $$s_{ij}^{Ba} = \frac{\partial B}{\partial a_{ij}}\bigg|_{A=A_0} = \begin{bmatrix} \frac{\partial b_{11}}{\partial a_{ij}} & \frac{\partial b_{12}}{\partial a_{ij}} & \cdots & \frac{\partial b_{1m}}{\partial a_{ij}} \\ \frac{\partial b_{21}}{\partial a_{ij}} & \frac{\partial b_{22}}{\partial a_{ij}} & \cdots & \frac{\partial b_{2m}}{\partial a_{ij}} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial b_{n1}}{\partial a_{ij}} & \frac{\partial b_{n2}}{\partial a_{ij}} & \cdots & \frac{\partial b_{nm}}{\partial a_{ij}} \end{bmatrix}_{A=A_0}$$

is the ij-th block of S^{BA},

(4.8c) $$s_{ij}^{xa}(t_0) = \frac{\partial x(t_0)}{\partial a_{ij}}\bigg|_{A=A_0} = \mathrm{col}\left(\frac{\partial x_1(t_0)}{\partial a_{ij}}, \ldots, \frac{\partial x_n(t_0)}{\partial a_{ij}}\right)\bigg|_{A=A_0}$$

is the ij-th block of $S^{xA}(t_0)$, and

(4.8d) $$e_j = \mathrm{col}(0, \ldots, \underbrace{0, 1, 0}_{j\text{-th entry}}, \ldots, 0)$$

and x_{0i} is the i-th entry of $x_0(t)$.

The solution of Eq. (4.7) is the set of vectors $s_{ij}^{xa}(t)$, $i, j = 1, \ldots, n$. These vectors give the sensitivity matrices of the first and the second form:

(4.9) $$T^{xa} = [s_{11}^{xa} s_{21}^{xa} \ldots s_{nn}^{xa}],$$

(4.10) $$S_k^{xA} = \begin{bmatrix} s_{11k}^{xa} & s_{12k}^{xa} & \cdots & s_{1nk}^{xa} \\ s_{21k}^{xa} & s_{22k}^{xa} & \cdots & s_{2nk}^{xa} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ s_{n1k}^{xa} & s_{n2k}^{xa} & \cdots & s_{nnk}^{xa} \end{bmatrix}$$

where s_{ijk}^{xa} is the k-th entry of the vector s_{ij}^{xa}. In this way the relationship between variations $\Delta A(t)$ and $\Delta x(t)$ has the form

(4.11) $$\Delta x(t) = T^{xa}(t)\Delta a(t),$$

(4.12) $$\Delta x(t) = [S_k^{xA}(t) \circ \Delta A(t)], \quad k = 1, \ldots, n,$$

where

(4.13) $$\Delta a = cs(\Delta A).$$

The relationship between the first form correlation matrices is as follows:

$$R_x^{(1)}(t_1, t_2) = T^{xa}(t_1) R_a^{(1)}(t_1, t_2) \big(T^{xa}(t_2)\big)^T, \tag{4.14}$$

where

$$R_x^{(1)}(t_1, t_2) = E\big(\Delta x(t_1) \Delta x^T(t_2)\big),$$
$$R_a^{(1)}(t_1, t_2) = E\big(\Delta a(t_1) \Delta a^T(t_2)\big),$$

and between the second form correlation matrices has the form

$$R_x^{(2)}(t_1, t_2) = [\mathscr{S}_{kl}^{xA}(t_1, t_2) \circ R_A^{(2)}(t_1, t_2)], \tag{4.15}$$

$k, l = 1, \ldots, n$; where

$$\mathscr{S}_{kl}^{xA}(t_1, t_2) = S_k^{xA}(t_1) \otimes S_l^{xA}(t_2) \tag{4.16}$$

and

$$R_x^{(2)}(t_1, t_2) = E\big(\Delta x(t_1) \otimes \Delta x(t_2)\big),$$
$$R_A^{(2)}(t_1, t_2) = E\big(\Delta A(t_1) \otimes \Delta A(t_2)\big).$$

Example 2. The system is described by Eq. (1.1), where

$$x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad u(t) = \delta(t),$$

and $x(t_0) = 0$.

The system matrix A

1. Has a deterministic variation ΔA

$$\Delta A = \begin{bmatrix} 0 & 0 \\ 10 & 0.2 \end{bmatrix}$$

with respect to the fixed value A_0:

$$A_0 = \begin{bmatrix} 0 & 1 \\ -100 & -2 \end{bmatrix}.$$

The fixed value $x_0(t)$ and the variation $\Delta x(t)$ are determined.

2. Has the random variation described by the second form correlation matrix:

$$R_A^{(2)} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 50 & -2 & -2 & 0.6 \end{bmatrix}$$

and the value A_0 as above. The mean value of $x(t)$ and its second form correlation matrix are determined.

In both cases it is assumed that matrices A and B do not depend on time and, additionally, that B and $x(t_0)$ do not depend upon A.

Since the fixed value of A in case 1 and the mean value of A in case 2 are both equal to A_0, then both the fixed value of $x(t)$ and its mean value are determined from the equation

$$\dot{x}_0(t) = A_0 x_0(t) + B\delta(t).$$

The solution of this equation is

$$x_0(t) = \begin{bmatrix} x_{01}(t) \\ x_{02}(t) \end{bmatrix} = 0.1005e^{-t} \begin{bmatrix} \sin(9.95t) \\ 10\sin(9.95t - 1.47) \end{bmatrix}.$$

Now, from Eq. (4.7) we determine s_{ij}^{xa} for $i, j = 1, 2$; since B and $x(t_0)$ do not depend on A, therefore $s_{ij}^{ba} = 0$, $s_{ij}^{xa}(t_0) = 0$ for $i, j = 1, 2$. Consequently, we obtain Eqs. (4.7) in the form

$$\dot{s}_{11}^{xa}(t) = A_0 s_{11}^{xa}(t) + x_{01}(t) \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

$$\dot{s}_{12}^{xa}(t) = A_0 s_{12}^{xa}(t) + x_{01}(t) \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

$$\dot{s}_{21}^{xa}(t) = A_0 s_{21}^{xa}(t) + x_{02}(t) \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

$$\dot{s}_{22}^{xa}(t) = A_0 s_{22}^{xa}(t) + x_{02}(t) \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

The solution of these equations is

$$s_{11}^{xa}(t) = \begin{bmatrix} s_{111}^{xa}(t) \\ s_{112}^{xa}(t) \end{bmatrix} = 0.0005076\, e^{-t} \begin{bmatrix} \sin\omega t - 99.5t\cos(\omega t + 1.47) \\ -100(\sin\omega t - 9.95t\cos\omega t) \end{bmatrix},$$

$$s_{12}^{xa}(t) = \begin{bmatrix} s_{121}^{xa}(t) \\ s_{122}^{xa}(t) \end{bmatrix} = 0.0005076 e^{-t} \begin{bmatrix} \sin\omega t - 9.95t\cos\omega t \\ -\sin\omega t - 99.5t\cos(\omega t - 1.47) \end{bmatrix},$$

$$s_{21}^{xa}(t) = \begin{bmatrix} s_{211}^{xa}(t) \\ s_{212}^{xa}(t) \end{bmatrix} = 0.05076 e^{-t} \begin{bmatrix} 0.98\sin\omega t - 9.95t\cos\omega t \\ \sin\omega t + 99.5t\cos(\omega t - 1.47) \end{bmatrix},$$

$$s_{22}^{xa}(t) = \begin{bmatrix} s_{221}^{xa}(t) \\ s_{222}^{xa}(t) \end{bmatrix} = 0.05076\, e^{-t} \begin{bmatrix} -0.01\sin\omega t - 0.995t\cos(\omega t - 1.47) \\ \sin\omega t - 9.95t\cos(\omega t + 0.20) \end{bmatrix},$$

where $\omega = 9.95$. From Eq. (4.10) we obtain matrices $S_1^{xA}(t)$ and $S_2^{xA}(t)$:

$$S_1^{xA}(t) = \begin{bmatrix} s_{111}^{xa}(t) & s_{121}^{xa}(t) \\ s_{211}^{xa}(t) & s_{221}^{xa}(t) \end{bmatrix},$$

$$S_2^{xA}(t) = \begin{bmatrix} s_{112}^{xa}(t) & s_{122}^{xa}(t) \\ s_{212}^{xa}(t) & s_{222}^{xa}(t) \end{bmatrix},$$

therefore, from Eq. (4.12) we have

$$\Delta x_1(t) = S_1^{xA}(t) \circ \Delta A = 10 s_{211}^{xa}(t) + 0.2\, s_{221}^{xa}(t)$$
$$= 0.05074\, e^{-t}(9.8\sin\omega t - 99.5t\cos\omega t - 0.199\cos(\omega t - 1.47)),$$

$$\Delta x_2(t) = S_2^{xA}(t) \circ \Delta t = 10\, s_{212}^{xa}(t) + 0.2\, s_{222}^{xa}(t)$$

$$= 0.05076\, e^{-t}(-9.8 \sin\omega t - 995\, t\cos(\omega t - 1.47) - 0.199\, t\cos(\omega t + 0.20)$$

and

$$\Delta x(t) = \begin{bmatrix} \Delta x_1(t) \\ \Delta x_2(t) \end{bmatrix}.$$

The graphs of the functions $\Delta x_1(t)$ and $\Delta x_2(t)$ are presented in Fig. 2.

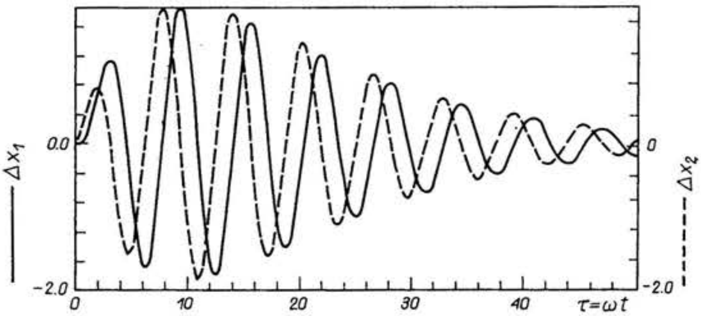

FIG. 2.

The correlation matrix $R_x^{(2)}(t_1, t_2)$ is computed from the formulas (4.15) and (4.16). We obtain

$$\mathscr{S}_{11}^{xA}(t_1, t_2) = S_1^{xA}(t_1) \otimes S_1^{xA}(t_2) =$$

$$= \left[\begin{array}{cccc|cccc} s_{111}^{xa}(t_1) & s_{111}^{xa}(t_2) & s_{111}^{xa}(t_1) & s_{121}^{xa}(t_2) & s_{121}^{xa}(t_1) & s_{111}^{xa}(r_2) & s_{121}^{xa}(t_1) & s_{121}^{xa}(t_2) \\ s_{111}^{xa}(t_1) & s_{211}^{xa}(t_2) & s_{111}^{xa}(t_1) & s_{221}^{xc}(t_2) & s_{121}^{xa}(t_1) & s_{211}^{xa}(t_2) & s_{121}^{xa}(t_1) & s_{221}^{xa}(t_2) \\ \hline s_{211}^{xa}(t_1) & s_{111}^{xa}(t_2) & s_{211}^{xa}(t_1) & s_{121}^{xa}(t_2) & s_{221}^{xa}(t_1) & s_{111}^{xa}(t_2) & s_{221}^{xa}(t_1) & s_{122}^{xa}(t_2) \\ s_{211}^{xa}(t_1) & s_{211}^{xa}(t_2) & s_{211}^{xa}(t_1) & s_{221}^{xa}(t_2) & s_{221}^{xa}(t_1) & s_{211}^{xa}(t_2) & s_{221}^{xa}(t_1) & s_{221}^{xa}(t_2) \end{array}\right],$$

therefore

$$R_{x11}(t_1, t_2) = \mathscr{S}_{11}^{Ax}(t_1, t_2) \circ R_A^{(2)} = 50\, s_{211}^{xa}(t_1)\, s_{221}^{xa}(t_2)$$

$$-2\, s_{211}^{xa}(t_1)\, s_{221}^{xa}(t_2) - 2\, s_{221}^{xa}(t_1)\, s_{211}^{xa}(t_2) + 0.6\, s_{221}^{xa}(t_1)\, s_{221}^{xa}(t_2).$$

Similarly, we obtain

$$R_{x12}(t_1, t_2) = \mathscr{S}_{12}^{xA}(t_1, t_2) \circ R_A^{(2)} = 50\, s_{211}^{xa}(t_1)\, s_{212}^{xa}(t_2) - 2\, s_{211}^{xa}(t_1)\, s_{222}^{xa}(t_2)$$

$$-2\, s_{221}^{xa}(t_1)\, s_{212}^{xa}(t_2) + 0.6\, s_{221}^{xa}(t_1)\, s_{222}^{xa}(t_2),$$

$$R_{x21}(t_1, t_2) = R_{x12}(t_2, t_1),$$

$$R_{x22}(t_1, t_2) = \mathscr{S}_{22}^{xA}(t_1, t_2) \circ R_A^{(2)} = 50\, s_{212}^{xa}(t_1)\, s_{212}^{xa}(t_2) - 2\, s_{212}^{xa}(t_1)\, s_{222}^{xa}(t_2)$$

$$-2\, s_{222}^{xa}(t_1)\, s_{212}^{xa}(t_2) + 0.6\, s_{222}^{xa}(t_1)\, s_{222}^{xa}(t_2),$$

and

$$R_x^{(2)}(t, t_2) = \begin{bmatrix} R_{x11}(t_1, t_2) \\ R_{x12}(t_1, t_2) \\ R_{x21}(t_1, t_2) \\ R_{x22}(t_1, t_2) \end{bmatrix}.$$

The graphs of the functions $R_{x11}(t, t)$, $R_{x12}(t, t) = R_{x21}(t, t)$ and $R_{x22}(t, t)$ are given in Fig. 3.

Examples where the first from correlation matrices are used can be found in [2].

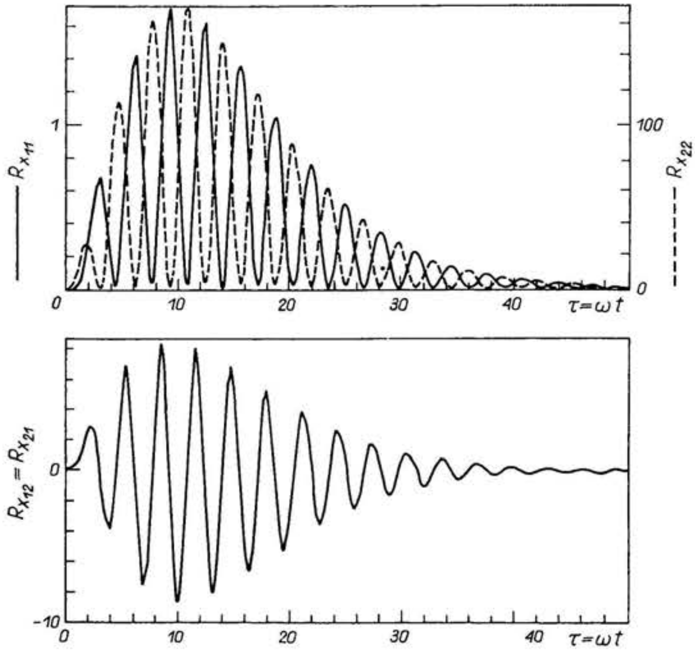

FIG. 3.

5. Conclusions

The paper has analysed a dynamic system with a white noise stochastic input as well as with a variable system matrix. The deterministric and random variations of the matrix have been considered. The analysis has been provided in two ways based on two representations of the matrix Taylor expansion [3] and [7], and two definitions of a correlation matrix [3]. It has been shown that in the case of a white noise stochastic input, when the new definition of a correlation matrix is applied, the correlation matrix of the output can be found as a solution of an ordinary linear differential equation. The previously existing definition of a correlation matrix in this case leads to a Riccati differential equation.

In the case of a variable system matrix two ways of solution have been presented as well. The method is applied to the analysis of the parametric sensitivity of mechanical system [2, 4] and to the analysis of data errors in computations of large systems [2, 5].

Appendix

Some notations and relationships

$E(.)$ mean value operator;

$E^{n\times n}_{n\times n}$ permutation matrix $N\times N$ dimensioned ($N = n^2$);

col(.) symbol of vector;

A^T transposition of a matrix A;

$\overline{A}$ complex conjugate matrix of A;

A^* complex conjugate transposition of A;

$cs(A)$ column transformation of a matrix $A, p\times q$: $cs(A) = \mathrm{col}(a_i)$; $i = 1, \ldots, q$, here a_i is the i-th column of A;

$A\circ B$ inner product of matrices $\underset{p\times q}{A}$ and $\underset{p\times q}{B}$: $A \circ B = \mathrm{tr}(AB^*)$, where $\mathrm{tr}(AB^*)$ is a trace of a matrix AB^*;

$A\otimes B$ direct (Kronecker) product of matrices $\underset{p\times q}{A}$ and $\underset{s\times t}{B}$: $A\otimes B = [a_{ij}B]$, $i = 1, \ldots, p$, $j = 1, \ldots, q$;

$A^{\otimes k}$ Kronecker power of a matrix A: $A^{\otimes k} = A\otimes A\otimes \ldots \otimes A$ (k factors), $A^{\otimes 0} = 1$, $A^{\otimes 1} = A$.

The following property of the direct product is valid [7]:

(A1) $$(AB)\otimes(CD) = (A\otimes C)(B\otimes D)$$

if the proper operations are possible;

$\dfrac{\partial A}{\partial B}$ — derivative of a matrix $\underset{p\times q}{A}$ with respect to a matrix $\underset{s\times t}{B}$

$$\frac{\partial A}{\partial B} = \left[\frac{\partial A}{\partial b_{kl}}\right], \quad k = 1, \ldots, s; \quad l = 1, \ldots, t,$$

where

$$\frac{\partial A}{\partial b_{kl}} = \left[\frac{\partial a_{ij}}{\partial b_{kl}}\right], \quad i = 1, \ldots, p, \quad j = 1, \ldots, q.$$

The following properties of the matrix derivative are valid:

(A2) $$\frac{d}{dt}(F\otimes G) = \frac{dF}{dt}\otimes G + F\otimes\frac{dG}{dt},$$

(A3) $$\frac{d}{dt}(FG) = \frac{dF}{dt}G + F\frac{dG}{dt}.$$

Matrix Taylor expansions:

a) from VETTER [7], for a matrix function $\underset{p\times q}{A}$ of a vector $b, s\times 1$:

$$A(b) = A(b_0) + \sum_{k=1}^{N}\frac{1}{k!}\left(\left.\frac{\partial\otimes^k A}{\partial b^{T\otimes k}}\right|_{b=b_0}(b-b_0)^{\otimes k}\otimes \underset{q}{I}\right) + R_{N+1},$$

where

$$R_{N+1} = \frac{1}{(N+1)!}\int_{b_0}^{b}\left(\frac{\partial\otimes^{(N+1)}A}{\partial z^{T\otimes(N+1)}}\right)(\underset{s}{I}\otimes(b-z)^{\otimes N}\otimes\underset{q}{I})(dz\otimes\underset{q}{I});$$

b) from GAWROŃSKI [3], for a matrix $\underset{p\times q}{A}$ of a matrix argument $\underset{s\times t}{B}$:

$$a_{ij}(B) = a_{ij}(B_0) + \sum_{k=1}^{N} \frac{1}{k!} \frac{\partial^{\otimes k} a_{ij}}{\partial B^{\otimes k}}\Big|_{B=B_0} \circ (B-B_0)^{\otimes k} + r_{ij}^{(N+1)},$$

$i = 1, \ldots, p,\ \ j = 1, \ldots, q$, where:

$$r_{ij}^{(N+1)} = \frac{1}{(N+1)!} \int_{B_0}^{B} \frac{\partial^{\otimes(N+1)} a_{ij}}{\partial Z^{(\otimes N+1)}} \circ d(B-Z)^{\otimes(N+1)}.$$

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Received October 22, 1975.