Archives of Mechanics • Archiwum Mechaniki Stosowanej • 29, 3, pp. 383—392, Warszawa 1977

# Investigation of a two-dimensional model of a micropolar continuum

K. BERGLUND (STOCKHOLM)

THE DIFFERENTIAL field equations of motion for a micropolar continuum are derived by constructing the difference equations of motion for a discrete lattice model and going to the continuum limit of a small lattice period. The model used consists of a two-dimensional quadratic lattice of pulleys, where central forces and torques are conveyed by central springs and crossed tangential rubber bands. It is found that the polar character of the resulting continuum manifests itself only in an antisymmetric part of the stress tensor, whereas the micropolar angular momentum and the couple stress turn out to be negligible. It is argued that this is a general feature of micropolar continua, modelled by ordinary classical-mechanical devices on the microlevel, e.g., composite materials.

Równania różniczkowe ruchu dla ośrodka mikropolarnego wyprowadzone z równań ruchu w formie różnicowej dla dyskretnego modelu siatkowego są poprawne w granicy dla ośrodka kontynualnego przy zmniejszeniu siatki. Stosowany model składa się z dwuwymiarowych kwadratowych sieci ścięgien, gdzie siły centralne i momenty skręcające są przenoszone przez centralne sprężyny i skrzyżowane, styczne gumowe ograniczniki. Wykazano, że polarny charakter powstałego ośrodka ciągłego przejawia się tylko w antysymetrycznej części tensora naprężenia, podczas gdy mikropolarny moment pędu i naprężenia momentowe są zaniedbywane. Okazuje się, że jest to ogólna cecha mikropolarnych ośrodków ciągłych modelowanych przez zwykłe klasyczne układy mechaniczne na poziomie mikro, np. materiałów kompozytowych.

Дифференциальные уравнения движения для микрополярной среды выведены из уравнения движения в разностной форме для дискретной сеточной модели, переходя в пределе к континуальной среде при уменьшении сетки. Применяемая модель состоит из двухмерной квадратной стеки струн, в которой центральные силы и скручивающие моменты переносятся через центральные пружины и скрещенные касательные резиновые ограничители. Показано, что полярный характер возникшей сплошной среды проявляется только в антисимметричной части тензора напряжений, так как микрополярным моментом импульса и моментным напряжением можно пренебречь. Оказывается, это общее свойство микрополярных сплошных сред, моделированных через обыкновенные классические механические системы на уровне микро, например, композитных материалов.

## 1. Introduction and summary

THE AIM of the present investigation is to illustrate to what extent a linear micropolar continuum, satisfying the usual field equations given, for example, by NOWACKI [1], can be obtained as a continuum limit of a discrete, classical-mechanical lattice model. This means that on the microlevel as described by the model the micropoles and their interactions are supposed to have the same order of magnitude as classical-mechanical systems of dimensions comparable to the lattice period as, for instance, in a composite material.

As a representative model of this kind we consider a two-dimensional system of pulleys, between which central forces and torqes are applied by means of springs and rubber bands. When in equilibrium the pulleys form a square lattice as shown in Fig. 1, there positions are indicated by indices $(p, q)$ in a coordinate system.

The cohesive interaction is created by the following mechanism indicated in Fig. 1:

A. Springs parallel to the lattice between each pulley and its nearest neighbours.

B. Diagonal springs between each pulley and its second nearest neighbours.

C. Crossed rubber bands between each pulley and its nearest neighbours.

It is seen from Fig. 1 that in order to calculate how a pulley $(p, q)$ is influenced by the others, only the eight adjacent pulleys have to be considered. The total force on a pulley $(p, q)$ is most easily obtained by examining the cases $A$, $B$ and $C$ separately and finally

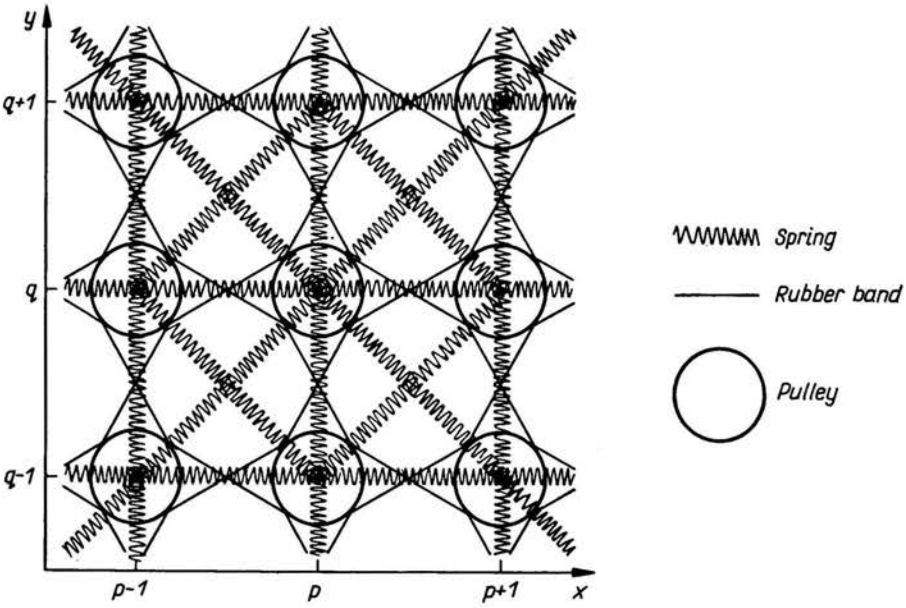


FIG. 1.

by adding the resulting expressions of forces and moments. As only small displacements from equilibrium are considered, only linear terms will be included.

In Sec. 2 we calculate the total resulting force, Eqs. (2.17) and (2.18), from springs and rubber bands, exerted on the pulley $(pq)$, giving the equations of motion for its centre of gravity, Eqs. (2.19) and (2.20).

In Sec. 3 we calculate the total resulting torque, Eq. (3.2), exerted solely by the rubber bands on the pulley $(p, q)$, which gives us the equation of angular momentum for this pulley, Eq. (3.3).

In Sec. 4 we discuss the character of the limiting process. The performance of the process gives the differential field equations for the micropolar continuum, i.e., for the balance of momentum, Eqs. (4.5) and (4.6), and angular momentum (4.7).

In Sec. 5 we compare our results with the current micropolar theory as described, for example, by NOWACKI [1]. It is seen that our resulting continuum has vanishing couple stress and vanishing micropolar angular momentum. In fact, terms of this kind have vanished at the limit procedure since they depend on higher powers of the lattice period than those which give the antisymmetric part of the stress tensor. This latter part is consequently the only remaining constitutive term in the balance of angular momentum and it has thus to balance by itself any applied body couple.

It may be noticed that the similar two-dimensional model with elastic rods conveying the interactions, studied by ASKAR and CAKMAK [2] and generalized to the three-dimensional case by TAUCHERT [3], show the same general properties as ours. In fact, an explicit assessment of the lattice period dependence of the constitutive constants in their terms for stress couple and micropolar angular momentum shows that these terms vanish at the limit process in the same way as ours. It seems reasonable that this is a general feature of all models where the microlevel can be modelled by classical-mechanical mechanism of the order of magnitude of the lattice period and can be made from materials having "ordinary" constitutive properties. In order to get a non-vanishing couple stress we must use a more sophisticated model where the ratio between the torque and the central force has a weaker dependence on the characteristic micropolar length.

## 2. Calculation of the resulting force

In the case of springs only the coordinates for the centre of the pulleys where the springs are assumed to be fastened will be taken into account. Thus a rotation of the pulleys does not influence the result.

The forces on $(p, q)$ from springs parallel to the lattice are calculated in the following way:

We study a representative pair, viz. pulley $(p, q)$ and $(p+1, q)$. The displacements are $\delta x_{p,q}$, $\delta y_{p,q}$ and $\delta x_{p+1,q}$, $\delta y_{p+1,q}$, respectively. As the displacements are small we restrict ourselves to linear terms of $\delta x$ and $\delta y$. The force on $(p, q)$ from $(p+1, q)$ will thus be

$$f_{x,(p,q),(p+1,q)} = K_1(d+\delta x_{p+1,q}-\delta x_{p,q}-h_1), \tag{2.1}$$

$$f_{y,(p,q),(p+1,q)} = K_1(\delta y_{p+1,q}-\delta y_{p,q})\left(1-\frac{h_1}{d}\right), \tag{2.2}$$

where $f_x, f_y$ are the $x$-component and $y$-component of the force respectively, $K_1$ the spring constant, $h_1$ the natural length of the springs and $d$ the distance between the centre of the pulleys at equilibrium.

In the same manner we obtain the force on $(p, q)$ from the three other pulleys $(p-1, q)$, $(p, q+1)$, $(p, q-1)$; the total forces from springs parallel to the lattice are

$$f^{(P)}_{x,(p,q)} = K_1\Big[(\delta x_{p+1,q}-\delta x_{p,q})-(\delta x_{p,q}-\delta x_{p-1,q}) + \left(1-\frac{h_1}{d}\right)\big((\delta y_{p,q+1}-\delta y_{p,q})-(\delta y_{p,q}-\delta y_{p,q-1})\big)\Big], \tag{2.3}$$

$$f^{(P)}_{y,(p,q)} = K_1\Big[(\delta y_{p,q+1}-\delta y_{p,q})-(\delta y_{p,q}-\delta y_{p,q-1}) + \left(1-\frac{h_1}{d}\right)\big((\delta y_{p+1,q}-\delta y_{p,q})-(\delta y_{p,q}-\delta y_{p-1,q})\big)\Big]. \tag{2.4}$$

The force on $(p, q)$ from diagonal springs is calculated from the representative pair $(p, q)$, $(p+1, q+1)$. The displacements are $\delta x_{p,q}$, $\delta v_{p,q}$ and $\delta x_{p+1,q+1}$, $\delta y_{p+1,q+1}$.

We obtain

$$(2.5)\qquad f_{x,(p,q),(p+1,q+1)} = K_2\left[d-\frac{h_2}{\sqrt{2}}+\left(1-\frac{h_2}{2d\sqrt{2}}\right)(\delta x_{p+1,q+1}-\delta x_{p,q}) + \frac{h_2}{2\sqrt{2}d}(\delta y_{p+1,q+1}-\delta y_{p,q}),\right.$$

where $K_2$ is the spring constant of the diagonal springs and $h_2$ the natural length of the diagonal springs, as the expression of the force in the direction of the $x$-axis and, similarly, for the $y$-component.

Thus the results of the total force from diagonal springs is

$$(2.6)\qquad f^{(D)}_{x,(p,q)} = K_2\left\{\left(1-\frac{1}{2\sqrt{2}}\frac{h_2}{d}\right)[(\delta x_{p+1,q+1}-\delta x_{p,q})-(\delta x_{p,q}-\delta x_{p-1,q-1}) + (\delta x_{p+1,q-1}-\delta x_{p,q})-(\delta x_{p,q}-\delta x_{p-1,q+1})] + \frac{1}{2\sqrt{2}}\frac{h_2}{d}(\delta y_{p+1,q+1}+\delta y_{p-1,q-1}-\delta y_{p+1,q-1}-\delta y_{p-1,q+1})\right\},$$

$$(2.7)\qquad f^{(D)}_{y,(p,q)} = K_2\left\{\left(1-\frac{1}{2\sqrt{2}}\frac{h_2}{d}\right)[(\delta y_{p-1,q+1}-\delta y_{p,q})-(\delta y_{p,q}-\delta y_{p+1,q-1}) + (\delta y_{p+1,q+1}-\delta y_{p,q})-(\delta y_{p,q}-\delta y_{p-1,q-1})] + \frac{1}{2\sqrt{2}}\frac{h_2}{d}(\delta x_{p+1,q+1}+\delta x_{p-1,q-1}-\delta x_{p-1,q+1}-\delta x_{p+1,q-1})\right\}.$$

The rubber bands are attached to the pulleys so that at equilibrium there is a right angle between the rubber band and the radius to the point where it is attached.

As the bands are attached in this manner we must consider a rotation $\delta\varphi$ of the pulleys as well as the position of their centres. When we calculate the force from the rubber bands a representative pair will be $(p, q)$, $(p+1, q)$. Consider a small displacement $\delta x_{p,q}$, $\delta y_{p,q}$ and a small rotation $\delta\varphi_{p,q}$ of pulley $(p, q)$, and an analogous displacement $\delta x_{p+1,q}$, $\delta y_{p+1,q}$ and an analogous rotation $\delta\varphi_{p+1,q}$ of pulley $(p+1, q)$. The displacements of the ends of the rubber bands due to the small rotation $\delta\varphi$ are assumed to take place in the direction of the tangent at equilibrium.

Let $l_1$ and $l_2$ be the length of the crossed rubber bands between $(p, q)$ and $(p+1, q)$. An expression of $l_1$ thus is

$$(2.8)\qquad l_1 = l_0\left[1+\frac{1}{d}(\delta x_{p+1,q}-\delta x_{p,q})+\frac{2b}{l_0^2}(\delta y_{p+1,q}-\delta y_{p,q})-\frac{bd}{l_0^2}(\delta\varphi_{p+1,q}+\delta\varphi_{p,q})\right],$$

where $b = rl_0/d$ and $r$ is the radius of a pulley and $l_0^2 = d^2-4r^2$.

Now we can write

$$(2.9)\qquad f_{x,(p,q),(p+1,q)} = K_3[(l_1-h_3)\cos\alpha_1+(l_2-h_3)\cos\alpha_2],$$

where $\alpha_1$ and $\alpha_2$ are the angles between Band 1 and the $x$-axis, and Band 2 and the $x$-axis, respectively. We can derive an expression for $\cos\alpha_1$ as follows:

$$(2.10)\qquad \cos\alpha_1 = \frac{1}{l_0}\left[\frac{l_0^2}{d}+\frac{4r^2}{d^2}(\delta x_{p+1,q}-\delta x_{p,q})-\frac{2b}{d}(\delta y_{p+1,q}-\delta y_{p,q})\right]$$

and similarly for $\cos\alpha_2$.

This gives us the following expression for the $x$-component of the force on $(p, q)$ from $(p, q+1)$ in the case of rubber bands:

$$f_{x,(p,q),(p+1,q)} = K_3\left[\frac{2l_0}{d}(l_0-h_3)+\left(2-\frac{8r^2h_3}{l_0d^2}\right)(\delta x_{p+1,q}-\delta x_{p,q})\right], \tag{2.11}$$

where $K_3$ is the spring constant for the rubber bands.

The $y$-component of the force from $(p+1, q)$ is

$$f_{y,(p,q),(p+1,q)} = K_3[(l_1-h_3)\sin\alpha_1-(l_2-h_3)\sin\alpha_2]. \tag{2.12}$$

An expression of $\sin\alpha_1$ is

$$\sin\alpha_1 = \frac{2b}{l_0}+\frac{l_0}{d^2}(\delta y_{p,q+1}-\delta y_{p,q})-\frac{2b}{l_0 d}(\delta x_{p,q+1}-\delta x_{p,q}) \tag{2.13}$$

and similarly for $\sin\alpha_2$.

Thus, the $y$-component of the force is

$$f_{y,(p,q),(p+1,q)} = K_3\left\{\left[\frac{8r^2}{d^2}+\frac{2l_0}{d^2}(l_0-h_3)\right](\delta y_{p+1,q}-\delta y_{p,q})-\frac{4b^2d}{l_0^2}(\delta\varphi_{p+1,q}+\delta\varphi_{p,q})\right\}. \tag{2.14}$$

The force from $(p-1, q)$, $(p, q+1)$, $(p, q-1)$ is obtained in a similar manner and the total force from the crossed rubber bands is

$$\begin{aligned} f^{(R)}_{x,(p,q)} = K_3\Bigg\{&\left[2-\frac{8r^2h_3}{l_0d^2}\right][(\delta x_{p+1,q}-\delta x_{p,q})-(\delta x_{p,q}-\delta x_{p-1,q})] \\ &+\left[\frac{8r^2}{d^2}+\frac{2l_0}{d^2}(l_0-h_3)\right][(\delta x_{p,q+1}-\delta x_{p,q})-(\delta x_{p,q}-\delta x_{p,q-1})]+\frac{4b^2d}{l_0^2}(\delta\varphi_{p,q+1}-\delta\varphi_{p,q-1})\Bigg\}, \end{aligned} \tag{2.15}$$

$$\begin{aligned} f^{(R)}_{y,(p,q)} = K_3\Bigg\{&\left[2-\frac{8r^2h_3}{l_0 d^2}\right][(\delta y_{p,q-1}-\delta y_{p,q})-(\delta y_{p,q}-\delta y_{p,q+1})] \\ &+\left[\frac{8r^2}{d^2}+\frac{2l_0}{d^2}(l_0-h_3)\right][(\delta y_{p+1,q}-\delta y_{p,q})-(\delta y_{p,q}-\delta y_{p-1,q})]+\frac{4b^2d}{l_0^2}(\delta\varphi_{p-1,q}-\delta\varphi_{p+1,p})\Bigg\}. \end{aligned} \tag{2.16}$$

The total force on $(p, q)$ is obtained in the following way:

$$f_{x,(p,q)} = f^{(P)}_{x,(p,q)}+f^{(D)}_{x,(p,q)}+f^{(R)}_{x,(p,q)}.$$

Inserting Eqs. (2.3), (2.6) and (2.15) we get

$$\begin{aligned} f_{x,(p,q)} = &\left[K_1+K_3\left(2-\frac{8r^2h_3}{l_0d^2}\right)\right][(\delta x_{p+1,q}-\delta x_{p,q})-(\delta x_{p,q}-\delta x_{p-1,q})] \\ &+\left[K_1\left(1-\frac{h_1}{d}\right)+K_3\left(\frac{8r^2}{l_0d^2}+\frac{2l_0}{d^2}\right)(l_0-h_3)\right][(\delta x_{p,q+1}-\delta x_{p,q})-(\delta x_{p,q}-\delta x_{p,q-1})] \\ &+K_2\Bigg\{\left(1-\frac{1}{2\sqrt{2}}\frac{h_2}{d}\right)[(\delta x_{p+1,q+1}-\delta x_{p+1,q})-(\delta x_{p+1,q}-\delta x_{p+1,q-1}) \\ &+(\delta x_{p-1,q+1}-\delta x_{p-1,q})-(\delta x_{p-1,q}-\delta x_{p-1,q-1})+2[(\delta x_{p+1,q}-\delta x_{p,q})-(\delta x_{p,q}-\delta x_{p-1,q})] \\ &+\frac{h_2}{2\sqrt{2}d}[(\delta y_{p+1,q+1}-\delta y_{p+1,q-1})-(\delta y_{p-1,q+1}-\delta y_{p-1,q-1})]\Bigg\} \\ &+K_3\frac{4b^2d}{l_0^2}(\delta\varphi_{p,q+1}-\delta\varphi_{p,q-1}). \end{aligned} \tag{2.17}$$

Similarly we obtain

$$(2.18)\qquad f_{y,(p,q)} = \left[K_1 + K_3\left(2 - \frac{8r^2h_3}{l_0 d^2}\right)\right][(\delta y_{p,q+1} - \delta y_{p,q}) - (\delta y_{p,q} - \delta y_{p,q-1})]$$

$$+\left[K_1\left(1 - \frac{h_1}{d}\right) + K_3\left(\frac{8r^2}{l_0 d^2} - \frac{2l_0}{d^2}\right)(l_0 - h_3)\right][(\delta y_{p+1,q} - \delta y_{p,q}) - (\delta y_{p,q} - \delta y_{p-1,q})]$$

$$+K_2\left\{\left(1 - \frac{1}{2\sqrt{2}}\frac{h_2}{d}\right)[(\delta y_{p+1,p+1} - \delta y_{p+1,q}) - (\delta y_{p+1,q} - \delta y_{p+1,q-1})]\right.$$

$$+(\delta y_{p-1,q+1} - \delta y_{p-1,q}) - (\delta y_{p-1,q} - \delta y_{p-1,q-1}) + 2[(\delta y_{p+1,q} - \delta y_{p,q}) - (\delta y_{p,q} - \delta y_{p-1,q})]$$

$$\left. + \frac{1}{2\sqrt{2}}\frac{h_2}{d}[(\delta x_{p+1,q+1} - \delta x_{p-1,q+1}) - (\delta x_{p+1,q-1} - \delta x_{p-1,q-1})]\right\}$$

$$-K_3\frac{4b^2d}{l_0^2}(\delta\varphi_{p+1,q} - \delta\varphi_{p-1,q}).$$

The equations of motion for the centre of gravity of pulley with the mass $m$ are thus

$$(2.19)\qquad m\frac{d}{dt^2}(\delta x_{p,q}) = f_{x,(p,q)},$$

$$(2.20)\qquad m\frac{d}{dt^2}(\delta y_{p,q}) = f_{y,(p,q)}.$$

## 3. Calculation of the resulting moment

In the present model torques are only conveyed by means of the rubber bands and since, in linear approximation, all the levers have the same magnitude $r$, we obtain

$$(3.1)\qquad M_{z,(p,q),(p+1,q)} = K_3[(l_1 - h_3) - (l_2 - h_3)]r = K_3 r(l_1 - l_2)$$

and similarly for the moments from $(p-1, q)$, $(p, q+1)$ and $(p, q-1)$.

In accordance to Eq. (3.1) and the three analogous expressions, the total moment is

$$(3.2)\qquad M_{z,(p,q)} = K_3\left\{\frac{4br}{l_0}[(\delta y_{p+1,q} - \delta y_{p-1,q}) - (\delta x_{p,q+1} - \delta x_{p,q-1})]\right.$$

$$\left. -2r^2(\delta\varphi_{p+1,q} + \delta\varphi_{p-1,q} + \delta\varphi_{p,q+1} + \delta\varphi_{p,q-1} + 4\delta\varphi_{p,q})\right\}.$$

Thus, the law of angular momentum for the pulley $(p, q)$ with the mass $m$ will be

$$(3.3)\qquad mj\frac{d^2}{dt^2}(\delta\varphi_{p,q}) = M_{z,(p,q)},$$

where $j$ is the moment of inertia per unit mass.

## 4. The continuum limit

Going to the zero limit with the distance ($d$) between the pulleys, as well as with the radius, masses, and radius of inertia of the pulleys, we obtain a two-dimensional micropolar continuum.

The limiting procedure will be done in such a way that

$$\sigma = m/d^2$$

will have a finite value in the limit, viz. the mass density per unit area of the two-dimensional continuum. The limiting procedure will also be such that the moment of inertia $j$ per unit mass (= the square of the radius of inertia) goes to zero as $d^2$, since we assume that the radius of inertia is of the order of magnitude $d$.

The spring constants have a finite magnitude since a spring, composed of a square lattice of small, parallel springs, is easily seen to have the same spring constant as each of the small springs.

The natural lengths of the springs are normally assumed to be of the same order of magnitude as $d$, so that a quotient of type $h/d$ has a finite value in the limit.

In the limiting procedure the calculations (2.19) and (2.20) will thus be divided with $d^2$ and are therefore transformed in the following field equations:

$$\sigma\ddot{u} = \lim_{d\to 0}[d^{-2}f_{x,(p,q)}], \tag{4.1}$$

$$\sigma\ddot{v} = \lim_{d\to 0}[d^{-2}f_{y,(p,q)}], \tag{4.2}$$

where we have put

$$u = \delta x_{p,q}, \quad v = \delta y_{p,q}.$$

After division with $d^2$ the equation of angular momentum (3.3) is transferred into the following field equation:

$$\lim_{d\to 0}\sigma j\ddot{\varphi} = \lim_{d\to 0}[d^{-2}M_{z,(p,q)}], \tag{4.3}$$

where we have put

$$\delta\varphi_{p,q} = \varphi.$$

If we assume $\ddot{\varphi}$ to be of the same order of magnitude as the angular velocity of the lattice, i.e., Cauchy's spin-tensor, the left hand side of Eq. (4.3) will disappear and the equation of angular momentum will be

$$\lim_{d\to 0}[d^{-2}M_{z,(p,q)}] = 0. \tag{4.4}$$

Passing to the limit we form difference quotients which are transformed in derivatives. In doing so we put

$$\Delta x = x_{p+1,q}-x_{p,q} = x_{p,q}-x_{p-1,q} = d,$$

$$\Delta y = y_{p,q+1}-y_{p,q} = y_{p,q}-y_{p,q-1} = d.$$

Performing this limit procedure on the law of momentum, Eqs (4.1) and (4.2), and the law of angular momentum, Eq. (4.4), we obtain

$$(4.5)\quad \sigma\ddot{u} = \left[K_1+K_2\left(2-\frac{h_2}{2\sqrt{2}}\right)+K_3\left(2-\frac{8r^2h_3}{d^2l_0}\right)\right]\frac{\partial^2 u}{\partial x^2}+\left[K_1\left(1-\frac{h_1}{d}\right)+K_2\left(2-\frac{h_2}{d\sqrt{2}}\right)\right.$$
$$\left.+K_3\left(\frac{8r^2}{d^2}+\frac{2l_0(l_0-h_3)}{d^2}\right)\right]\frac{\partial^2 u}{\partial y^2}+\sqrt{2}K_2\frac{h_2}{d}\left[\frac{\partial^2 v}{\partial y\,\partial x}\right]+K_3\frac{8r^2}{d^2}\frac{\partial\varphi}{\partial y},$$

$$(4.6)\quad \sigma\ddot{v} = \left[K_1+K_2\left(2-\frac{h_2}{d\sqrt{2}}\right)+K_2\left(2-\frac{8r^2h_3}{d^2l_0}\right)\right]\frac{\partial^2 v}{\partial y^2}+\left[K_1\left(1-\frac{h_1}{d}\right)+K_2\left(2-\frac{h_2}{d\sqrt{2}}\right)\right.$$
$$\left.+K_3\left(\frac{8r^2}{d^2}+\frac{2l_0(l_0-h_3)}{d^2}\right)\right]\frac{\partial^2 v}{\partial x^2}+\sqrt{2}K_2\frac{h_2}{d}\left(\frac{\partial^2 u}{\partial x\,\partial y}\right)-K_3\frac{8r^2}{d^2}\left(\frac{\partial\varphi}{\partial x}\right),$$

$$(4.7)\quad K_3\frac{8r^2}{d^2}\left[\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}-2\varphi\right]=0.$$

Equations (4.5) and (4.6) have the form ($u_1 = u$, $u_2 = v$, $u_3 = 0$, $\varphi_1 = 0$, $\varphi_2 = 0$, $\varphi_3 = \varphi$):

$$(4.8)\quad \sigma\ddot{u}_1 = Pu_{1,11}+Qu_{1,22}+Ru_{2,12}+S\varphi_{3,2},$$

$$(4.9)\quad \sigma\ddot{u}_2 = Qu_{2,11}+Pu_{2,22}+Ru_{1,12}-S\varphi_{3,1},$$

where the expressions for $P, Q, R$ and $S$ are given directly in Eqs. (4.5) and (4.6).

Isotropic equations must have the form

$$(4.10)\quad \sigma\ddot{u}_k = Au_{k,rr}+Bu_{r,kr}-C\varepsilon_{krs}\varphi_{r,s},$$

where $u_k$ and $\varphi_k$ do not depend on $x_3$ and $u_3 = 0$, $\varphi_1 = \varphi_2 = 0$.

A comparison with Eqs. (4.8) and (4.9) give the following conditions for isotropy:

$$(4.11)\quad Q+R = P.$$

Inserting values for $Q$, $R$ and $P$, we get

$$(4.12)\quad K_2 = \frac{h_1}{\sqrt{2}h_2}K_1+\frac{\sqrt{2}h_3(l_0^2-4r^2)}{h_2dl_0}K_3.$$

The law of momentum now takes the form

$$(4.13)\quad \sigma\ddot{u}_k = Qu_{k,rr}+Ru_{r,kr}+S\varepsilon_{krs}\varphi_{s,r}$$

and the law of angular momentum takes the form

$$(4.14)\quad S(\varepsilon_{klm}u_{m,l}-2\varphi_k) = 0.$$

## 5. Comparison with current micropolar theory

The equations for a general isotropic micropolar medium are, as seen for example, in NOWACKI [1],

$$(5.1)\quad \sigma\ddot{u}_k = (\lambda+\mu-\alpha)u_{l,kl}+(\mu+\alpha)u_{k,ll}+2\alpha\varepsilon_{kim}\varphi_{m,l},$$

$$(5.2)\quad \sigma j\ddot{\varphi}_k = (\beta+\gamma-\varepsilon)\varphi_{l,kl}+(\gamma+\varepsilon)\varphi_{k,ll}+2\alpha(\varepsilon_{klm}u_{m,l}-2\varphi_k),$$

where the constants $\lambda$, $u$, $\alpha$ enter into the stress-tensor and $\beta$, $\gamma$, $\varepsilon$ enter into the couple-stress-tensor.

A comparison with Eqs. (4.13) and (4.14) gives

$$\lambda+\mu-\alpha = R = \sqrt{2}K_2h_2d^{-1}, \tag{5.3}$$

$$\mu+\alpha = Q = P-R = K_1+K_2[2-3h_2(\sqrt{2}d)^{-1}]+K_3(2-8r^2h_3d^{-2}l_0^{-1}), \tag{5.4}$$

$$\alpha = \frac{S}{2} = K_3 4r^2d^{-2}, \tag{5.5}$$

$$j = 0, \gamma+\varepsilon = 0, \tag{5.6}$$

$\beta, \gamma-\varepsilon$ have no sense in 2 dimensions.

The inequalities

$$\begin{aligned} &0 \leqslant \alpha, \\ &0 \leqslant \mu, \\ &0 \leqslant 3\lambda+2\mu, \end{aligned} \tag{5.7}$$

which are the necessary and sufficient conditions for the medium to have a non-negative internal energy, are seen to be satisfied for values which have a reasonable order of magnitude of the occurring constants.

The comparable two-dimensional model with elastic rods instead of springs and rubber bands as studied by ASKAR and CALMAK [2] seems to have the same properties as ours.

If we study the entering coefficients in the expression of the strain energy in their model, we observe that it does not include any terms of type $\psi_{,x}^2$, $\psi_{,y}^2$, $\psi_{,xx}$, $\varphi_{,yy}$, because the coefficients which belong to these terms have an order of magnitude which is $a^2$ times the order of magnitude of the other entering coefficients, where $a$ is the distance between the nearest masses, and thus vanish in the limit. The result is that the couple stress is negligible.

A similar model generalized to three dimensions by TAUCHERT [3] also exhibits the same properties and, consequently, has a negligible couple stress.

It seems to be the case that if there is a dependence between the force and the torque of the same kind as in these models, viz. the torque on a micropole is a force having the same order of magnitude as the interacting forces times a lever which has the same order of magnitude as the lattice period, we can never get a non-vanishing couple stress. It is of no avail with a higher gradient theory because the terms of higher order vanish for the same reason as above.

If, in our model, there are also antisprings i.e., springs which have a negative spring constant together with the parallel springs, they should have the effect that the interacting forces will have a smaller order of magnitude than the force which gives the torque. In that case we would be able to get a non-vanishing couple stress. The same effect might be obtainable by electric or magnetic forces.

In conclusion the model studied here is linear, simple approximation gives a polar continuum theory with negligible angular momentum for the micropoles and with negligible couple stress. The polar properties are thus seen to show up only as an antisymmetric part of the stress tensor.

By introducing another kind of model which does not have any dependence between the interacting force and moment or at least has a weaker dependence than the models discussed above, it is possible to get a non-vanishing couple stress.

## Acknowledgment

I am indebted to Professor Stig HJALMARS for suggesting to me the topic of this investigation which is based on an idea originating from him and Professor Olof BRULIN. Without his unfailing help this work could not have been completed. I also wish to thank Professor Olof BRULIN for his readiness in giving me advice and lending me assistance.

## References

1. W. NOWACKI, *Theory of micropolar elasticity*, Springer, Wien, 1970.
2. A. ASKAR and A. S. CAKMAK, Int. J. Engng. Sci., **6**, 583, 1968.
3. T. R. TAUCHERT, Rec. Adv. Engng. Sci., **5**, 325, 1970.

THE ROYAL INSTITUTE OF TECHNOLOGY,
STOCKHOLM, SWEDEN.

*Received July* 17, 1975.