Archives of Mechanics • Archiwum Mechaniki Stosowanej • 29, 3, pp. 445–458, Warszawa 1977

On the hardening of soils(*)

R. NOVA (MILAN)

AN INCREMENTAL approach to the elastoplastic hardening of soils is presented. It is shown that in general the plastic strain increments depend on volumetric and shear strain previously experienced by the material specimen. The model presented here can be reduced to classical Cam Clay, but the choice of appropriate parameters leads to a better interpretation of experimental results. In particular the characteristic "hook" exhibited by the stress path of sand specimens in undrained conditions is predicted and realistic relations are obtained in constant hydrostatic pressure tests.

Przedstawiono przyrostową teorię sprężysto-plastycznego wzmocnienia ośrodków sypkich. Wykazano, że przyrosty odkształcenia plastycznego zależą na ogół od odkształceń postaciowych i objętościowych doznanych uprzednio przez próbkę materiału. Model przedstawiony w tej pracy może być sprowadzony do klasycznego modelu Cam Claya, lecz wybór właściwych parametrów prowadzi tu do lepszej interpretacji wyników doświadczalnych. W szczególności przewidziano charakterystyczny "haczyk" jaki wykazuje droga naprężeń dla próbek wilgotnego piasku i otrzymano realistyczne związki w doświadczeniach przy stałym ciśnieniu hydrostatycznym.

Представлена теория в приростах упруго-пластического упрочнения сыпучих сред. Показано, что приросты пластической деформации зависят в общем от деформаций сдвига и объемных деформаций, испытываемых раньше образцом материала. Модель, представленная в этой работе, может быть сведена к классической модели Кам Клея, но подбор правильных параметров приводит здесь к лучшей интерпретации экспериментальных результатов. В частности, предвиден характеристический „крюк'', которым обладает путь напряжений для образцов водонасыщенного песка и получены реальные соотношения в экспериментах при постоянном гидростатическом давлении.

Notation

C, D constants,
d dilatancy,
f yield locus,
g plastic potential,
G elastic shear modulus,
H hardening modulus,
K elastic bulk modulus,
p hydrostatic effective pressure,
p_c consolidation pressure,
p_u hidden variable,
q stress deviator,
V_0 initial volume,
$\dot{v}, \dot{v}^e, \dot{v}^p$ total, elastic, plastic volumetric strain rate,
$\dot{W}$ dissipated power,

(*) Paper presented at the Euromech Colloqium 84 on "Mechanics of Grannular Materials", Warsaw, July 1976.

$\dot{\varepsilon}, \dot{\varepsilon}^e, \dot{\varepsilon}^p$ total, elastic, plastic shear-strain rate,
ε_{ij} strain tensor,
η stress ratio,
Λ positive scalar,
λ, k virgin consolidation and swelling parameters,
k_i, χ_i hidden variables,
M critical state stress ratio,
σ_{ij} stress tensor.

1. Introduction

HISTORICALLY the behaviour of soil has been described in different ways depending on which purposes one had to fit. If the main problem was to guarantee the overall stability of an earth mass, the soil was idealised as a rigid, perfectly plastic material, and the classical theory of plasticity was used to predict the safety factor with respect to the global collapse of the system. Following the methods of the theory of plasticity, useful calculations of upper and lower bounds of the safety factor were performed (PRANDTL (1930), TERZAGHI (1943), DRUCKER and PRAGER (1952)) and in more recent years the Sokolovskii method of characteristics has been applied to evaluate solutions of complex problems (SOKOLOVSKII (1965)). On the other hand, when the major interest of the desinger was to avoid excessive settlements, the soil was idealised as a linearly elastic material and the well-known solutions of the theory of elasticity applied to predict settlements. Recently, more sophisticated analyses have been performed using a variable shear modulus (KONDNER (1963), KONDNER and ZELASKO (1963)), but still maintaining the hypothesis of an elastic behaviour, albeit nonlinear.

Unfortunately, both approaches are not completely satisfactory. The first disregards the influence of deformations on the collapse load. In particular, the dilatation properties of the soil are completely ignored. The second can give only a rough estimate of settlements because of the many restrictive hypotheses implied. In fact, some irrecoverable deformations take place from the very beginning of the loading process in most practical cases. This implies that not only the behaviour is non-linear, but also the direction of the strain increment is mainly determined by the state of stress and not by the stress increment, as in the elastic theory. This fact is not essential in cases in which only one parameter is representative of the state of stress, say the bending moment in a section of a beam, but is of importance where domains of allowable stresses ought to be considered, as practically in every Soil Mechanics problem.

To take account of this fact, many elastoplastic models for soil behaviour have been proposed. Among the most successful are Cam Clay (SCHOFIELD and WROTH (1968)) and more recently the model of LADE and DUNCAN for sands (1975). In both, consideration is given to the expressions of the plastic potential and of the yielding surface, whilst the definition of the hardening function is somewhat less convincing. In Cam Clay Schofield and Wroth assume that the volume is a function of the state of stress and of the loading history. They can define a state boundary surface that encloses all the possible states of the material and use it to evaluate the strain history. To evaluate the shear strain they integrate the shear strain rates along the stress path. Lade and Duncan assume that the dissipated work is a hyperbolic function of the shear strain. The strain increments are obtained by derivations.

The strain rates play a key role in an incremental theory of plasticity. If they are known, once the stress path is fixed it is possible to determine the total strain history step by step. A theory that is founded on the definition of the strain increments rather than on integral relations seems to be a more logical one. In this work, a general method to determine the expressions of the strain increments is put forward. Some simple assumptions will enable us to predict strains qualitatively close to reality in various conventional tests. Stress paths and pore pressures will also be predicted in an undrained test.

2. Strain increments

The following derivation is well known in metal plasticity. We shall assume that the material exhibits a work hardening behaviour, at least in the region in which we are concerned with. We shall suppose the existence of a family of plastic potentials that governs the direction of the strain increments, and of a family of yield loci that evolves as the soil hardens. Moreover, the soil will be considered as virgin at the first application of a load, which means that it will exhibit some irrecoverable deformation from the very beginning of the loading process.

Instead of working in the traditional space of tensors σ_{ij}, ε_{ij} it is extremely useful to limit ourselves to the "triaxial" plane defined for the first time by SCHOFIELD (1959). We thus define the following quantities:

$$p = \frac{\sigma_1' + 2\sigma_3'}{3}, \quad q = \sigma_1' - \sigma_3', \quad \eta = \frac{q}{p}, \tag{2.1}$$

$$\dot{v} = \dot{\varepsilon}_1 + 2\dot{\varepsilon}_3, \quad \dot{\varepsilon} = (\dot{\varepsilon}_1 - \dot{\varepsilon}_3). \tag{2.2}$$

A dash indicates effective stresses, a superimposed dot increments of strain or stress. The indices p and e will mean plastic and elastic strain, respectively:

$$\dot{v} = \dot{v}^e + \dot{v}^p, \quad \dot{\varepsilon} = \dot{\varepsilon}^e + \dot{\varepsilon}^p. \tag{2.3}$$

From the definitions (2.1), (2.2), we see that p and q are the effective hydrostatic pressure and the deviator of stress respectively, whilst $\dot{v}$ and $\dot{\varepsilon}$ are the corresponding volumetric and shear strain increments. Volumetric strains are taken positive in compression. The coefficient appearing in the definition of $\dot{\varepsilon}$ is necessary to express the dissipated power $\dot{W}$ as:

$$\dot{W} = \sigma_{ij}\dot{\varepsilon}_{ij} = p\dot{v}^p + q\dot{\varepsilon}^p. \tag{2.4}$$

For this simple derivation see SCHOFIELD and WROTH (1968).

In general we can define a plastic potential as

$$g(p, q, v, \varepsilon, k_i) = 0, \tag{2.5}$$

where k_i constitute a set of parameters depending on the previous history of the material. They are generally known as hidden variables and are functions of the state of strain. From the definition of plastic potential it follows that

$$\dot{v}^p = \Lambda \frac{\partial g}{\partial p}, \tag{2.6}$$

$$\dot{\varepsilon}^p = \Lambda \frac{\partial g}{\partial q}, \tag{2.7}$$

where Λ is a positive scalar.

We can then define the dilatancy d as

$$d = \frac{\dot{v}^p}{\dot{\varepsilon}^p} = \frac{\partial g/\partial p}{\partial g/\partial q}. \tag{2.8}$$

In a similar way one can express the yielding surface as

$$f(p, q, v, \varepsilon, \chi_i) = 0, \tag{2.9}$$

where χ_i constitute another set of hidden variables. Let us assume that the yielding surface can be expressed as a family of curves in the p, q plane depending on a single parameter p_u. This latter assumption is common to Cam Clay and the Lade Duncan model. It is not strictly necessary and what follows is valid also if the yield locus depends on more than one hidden variable, but this complicates the algebra without any appreciable result. Moreover, suppose that both the plastic potential and the yield locus do not depend on the stress path and therefore on the state of strain. Many experimental data are in accordance with this assumption (e.g. ROWE (1972), SCHOFIELD and WROTH (1968), LADE and DUNCAN (1975), POOROOSHASB (1971), TATSUOKA and ISHIHARA (1974)).

During any loading process in which plastic deformations occur, the yielding surface is constantly activated and therefore its increment $\dot{f}$ is null

$$\dot{f} = \frac{\partial f}{\partial p}\dot{p} + \frac{\partial f}{\partial q}\dot{q} + \frac{\partial f}{\partial p_u}\dot{p}_u = 0. \tag{2.10}$$

The increment $\dot{p}_u$ is a function of the plastic deformations and thus we may write

$$\dot{p}_u = \frac{\partial p_u}{\partial v^p}\dot{v}^p + \frac{\partial p_u}{\partial \varepsilon^p}\dot{\varepsilon}^p \tag{2.11}$$

and from the definitions (2.6), (2.7)

$$\dot{p}_u = \Lambda\left(\frac{\partial p_u}{\partial v^p}\frac{\partial g}{\partial p} + \frac{\partial p_u}{\partial \varepsilon^p}\frac{\partial g}{\partial q}\right). \tag{2.12}$$

Substituting in Eq. (2.10) we get

$$\Lambda = -\frac{\dfrac{\partial f}{\partial p}\dot{p} + \dfrac{\partial f}{\partial q}\dot{q}}{\dfrac{\partial f}{\partial p_u}\left(\dfrac{\partial p_u}{\partial v^p}\dfrac{\partial g}{\partial p} + \dfrac{\partial p_u}{\partial \varepsilon^p}\dfrac{\partial g}{\partial q}\right)}. \tag{2.13}$$

The plastic strain increments can then be determined through Eqs. (2.6), (2.7) once the yielding surface, the plastic potential, the stress increments and finally the dependence of p_u on the state of strain are known. The elastic increments $\dot{v}^e$ and $\dot{\varepsilon}^e$ are easily found once the elastic properties of the material are known. If K is the bulk modulus and G the shear modulus, it follows from Eqs. (2.1), (2.2), (2.3) that

$$(2.14) \qquad \dot{v}^e = \frac{\dot{p}}{K}, \quad \dot{\varepsilon}^e = \frac{\dot{q}}{3G}.$$

The history of strain as a function of the stress path can now be determined by integration.

Several plastic potentials and yielding surfaces have been presented in the literature which are suitable for the former representation. The aim of the following section is to determine a simple relation between p_u and the state of strain.

3. Hardening parameters

In Eq. (2.13) two quantities which have to be evaluated experimentally appear. The first, $\partial p_u/\partial v^p$, is an index of the evolution of the parameter p_u and, consequently, of the yielding surface with the variation of volume in a process without plastic shear strain, i.e. an isotropic compression. In fact, it is widely accepted that under an all round pressure the only occurring deformation is volumetric. The parameter p_u can be any convenient value of the hydrostatic pressure characterizing the current yield locus. Let us choose as p_u the pressure corresponding to the intersection of the current yield locus with the line $\eta = M$ along which the critical state is attained as shown in Fig. 1. On this line no plastic volume change is possible.

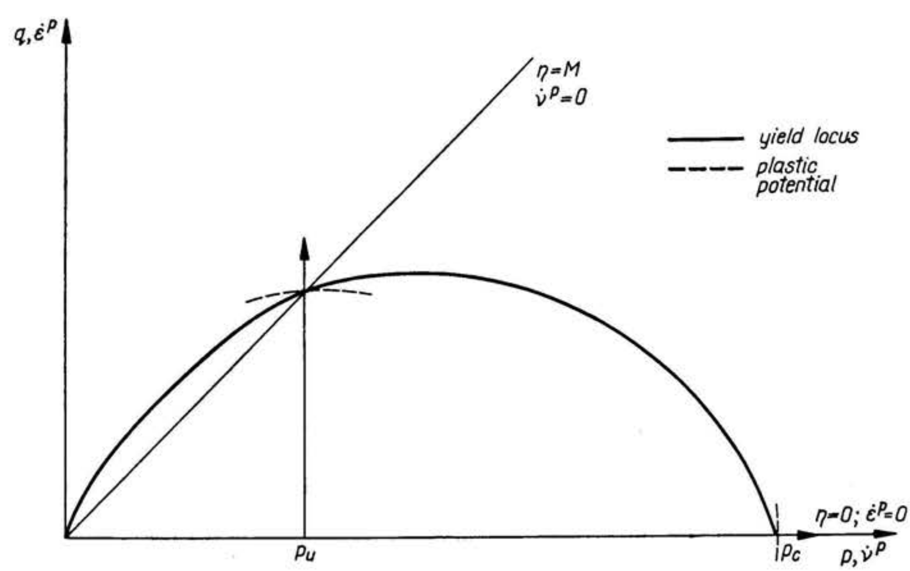

FIG. 1. Yield locus in p/q plane.

Suppose we test a specimen along a $q = 0$ stress path. A reasonable assumption, confirmed by experimental data is that

$$\dot{v}^p \Big|_{\eta=0} = C \frac{\dot{p}_u}{p_u}, \tag{3.1}$$

where C is a constant. This result follows when we assume the usual logarithmic relation between the plastic volumetric strain and the pressure p_c—see Fig. 1 — (SCHOFIELD and WROTH (1968)) and the dependence of the yield locus on the single parameter p_u. In this way the ratio p_c/p_u is constant and Eq. (3.1) follows. From this equation we can write

$$\frac{\partial p_u}{\partial v^p} = \frac{p_u}{C}. \tag{3.2}$$

To determine $\partial p_u/\partial \varepsilon^p$ we have to perform a test along the $\eta = M$ line. In fact, if the plastic potential is independent of the stress path, then, along this line, $\dot{v}^p = 0$ also during a hardening process, provided this is possible. Assume that

$$\dot{\varepsilon}^p \Big|_{\eta=M} = \frac{C}{D} \frac{\dot{p}_u}{p_u} \tag{3.3}$$

such that

$$\frac{\partial p_u}{\partial \varepsilon^p} = \frac{D}{C} p_u, \tag{3.4}$$

where D is another constant. Some η constant tests performed by NAMY (1970) on normally consolidated clays support this hypothesis.

Equation (2.13) now becomes

$$\Lambda = -C \frac{\dfrac{\partial f}{\partial p}\dot{p} + \dfrac{\partial f}{\partial q}\dot{q}}{\dfrac{\partial f}{\partial p_u}\left(\dfrac{\partial g}{\partial p} + D\dfrac{\partial g}{\partial q}\right)p_u}. \tag{3.5}$$

4. A simple example: an undrained test on Cam Clay

Equation (3.5) holds for any kind of flow rule, associated or non-associated. To show which are the implications of this kind of approach let us confine ourselves to the simple case of an associated flow rule. A very useful one to be treated analytically is the original Cam Clay. In this model

$$f \equiv g = q + M_p\left(\ln \frac{p}{p_u} - 1\right) = 0, \tag{4.1}$$

$$C = \lambda - k, \tag{4.2}$$

$$D = 0. \tag{4.3}$$

From Eq. (4.1) and the definition (2.8)

$$\frac{\partial f}{\partial q} = 1, \tag{4.4}$$

$$\frac{\partial f}{\partial p} = M \ln \frac{p}{p_u} = d, \tag{4.5}$$

$$\frac{\partial f}{\partial p_u} = -M \frac{p}{p_u}. \tag{4.6}$$

Thus Eq. (3.5) can be written as

$$\Lambda = (\lambda - k) \frac{d\dot{p} + \dot{q}}{Mpd}. \tag{4.7}$$

In Cam Clay

$$\dot{v}^e = k \frac{\dot{p}}{p}, \quad \dot{\varepsilon}^e = 0 \tag{4.8}$$

Then, in an undrained test ($\dot{v} = 0$)

$$\dot{v}^p = -k \frac{\dot{p}}{p} \tag{4.9}$$

and, from Eqs. (4.7), (2.6)

$$\dot{v}^p = -k \frac{\dot{p}}{p} = \left(d + \frac{\partial q}{\partial p} \right) \frac{\lambda - k}{M} \frac{\dot{p}}{p}. \tag{4.10}$$

Taking account of the fact that $d = M - \frac{q}{p}$, we get

$$\frac{\partial q}{\partial p} = \frac{q}{p} - \frac{M\lambda}{\lambda - k}. \tag{4.11}$$

Integrating, we have the stress path equation

$$q = \frac{M\lambda}{\lambda - k} p \ln \frac{p_c}{p} \tag{4.12}$$

and we can calculate the shear strains

$$\varepsilon = \int_0^{\varepsilon} \dot{\varepsilon}^p = \int_0^{p} \frac{-k}{p_d} \dot{p}. \tag{4.13}$$

From Eqs. (4.12), (4.13) we finally get

$$\frac{\eta}{M} = 1 - e^{-\frac{M\varepsilon}{k\left(1 - \frac{k}{\lambda}\right)}}. \tag{4.14}$$

This solution coincides with that presented by Schofield and Wroth expressed in terms of volumetric strain instead of volume. This means that the constants λ and k used here correspond to those used in Cam Clay divided by the initial volume V_0.

It is of interest to see what happens if D is taken as a positive constant. Equations (4.12) and (4.14) become respectively

$$\ln\frac{p}{p_c} = -\frac{\lambda-k}{\lambda}\left(\frac{\eta}{M}+\frac{k}{\lambda}\frac{D}{M}\ln\left(1-\frac{\eta/M}{1-\frac{k}{\lambda}\frac{D}{M}}\right)\right), \tag{4.15}$$

$$\frac{\eta}{M} = \left(1+\frac{k}{\lambda}\frac{D}{M}\right)\left(1-e^{-\frac{M\varepsilon}{k\left(1-\frac{k}{\lambda}\right)}}\right). \tag{4.16}$$

If we imagine that a sand can be modelled in this way, we note that much experimental evidence can be fitted better by taking $D > 0$. In particular, the characteristic "hook"

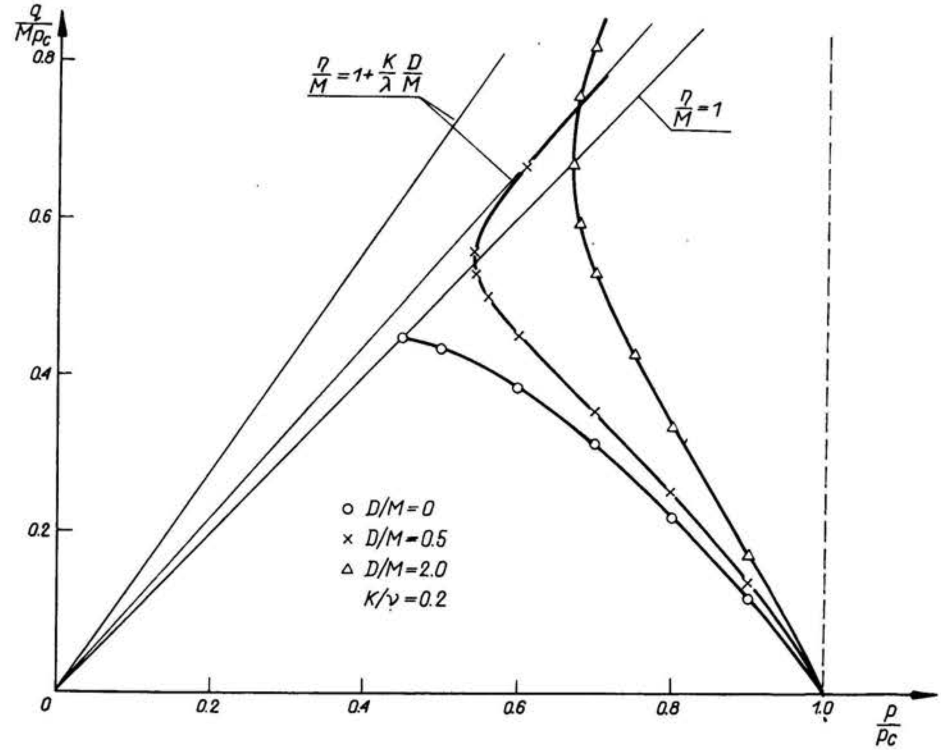

FIG. 2. Undrained test stress paths,
– – – drained stress path and total stress path.

exhibited by the stress path of sand specimens in undrained condition is predicted, as we can see in Fig. 2 where stress paths corresponding to different values of D/M are presented. We see how a low value of D/M can model loose sand, whilst a high value of D/M can model dense sand. The same conclusion holds for Figs. 3 and 4 in which pore pressures and stress ratios are plotted against shear strains.

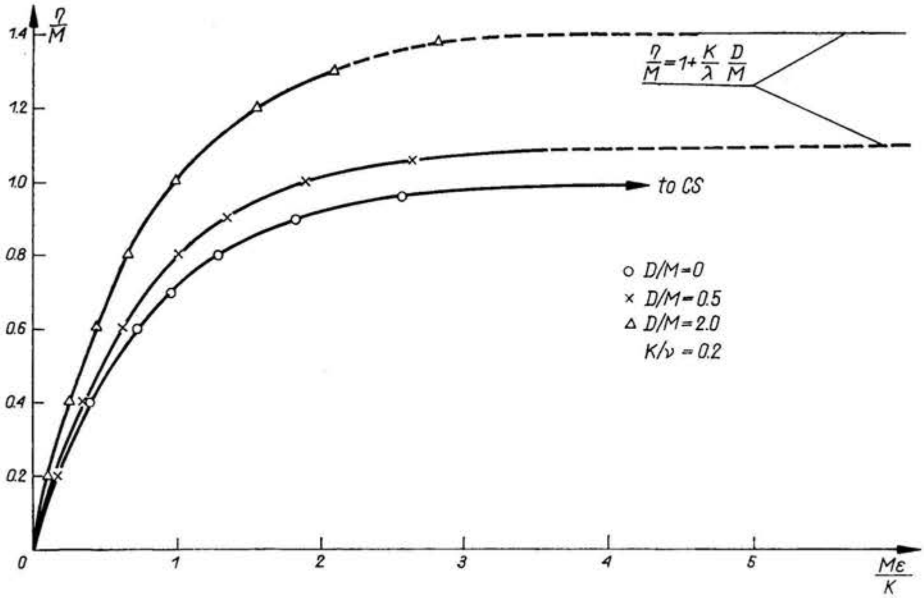

FIG. 3. Undrained test-stress strain law.

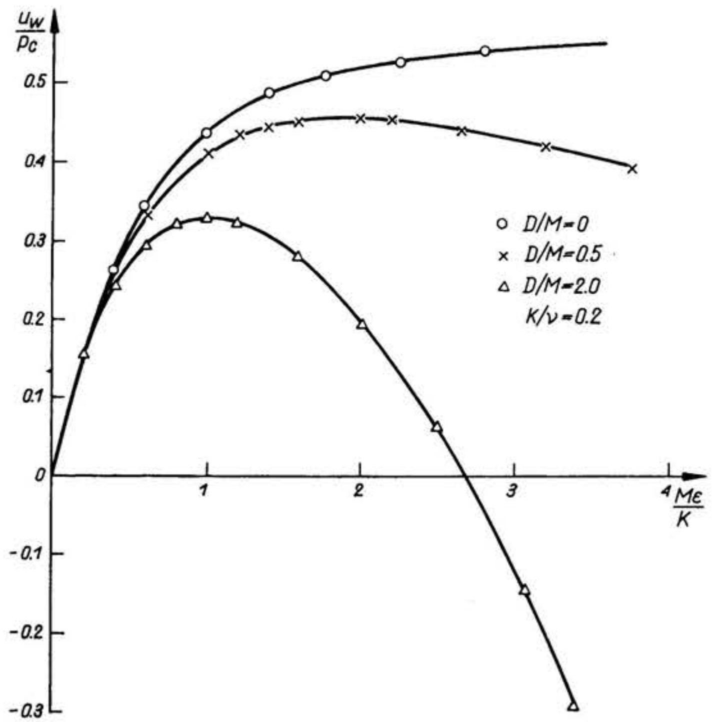

FIG. 4. Undrained test pore pressures.

5. Hardening-softening transition

In the theory of elastoplasticity allowing for hardening or softening, Eq. (2.13) is often written as

$$\Lambda = \frac{1}{H}\frac{\partial f}{\partial \sigma_{ij}}\dot{\sigma}_{ij}, \tag{5.1}$$

where H is referred to as "hardening modulus". $H > 0$ is said to characterize hardening, $H = 0$ perfectly plastic, $H < 0$ softening behaviour, respectively. Comparing Eq. (5.1) with Eq. (2.13), we see that

$$H = -\frac{\partial f}{\partial p_u}\left(\frac{\partial p_u}{\partial v^p}\frac{\partial g}{\partial q} + \frac{\partial p_u}{\partial \varepsilon^p}\frac{\partial g}{\partial q}\right) \tag{5.2}$$

or, if we accept Eqs. (3.2), (3.4), (4.1), (4.2),

$$H = \frac{M}{\lambda - k}p(d+D). \tag{5.3}$$

If D is positive, hardening is still possible when d is negative until the failure locus is reached.

Therefore, dilatation can occur during hardening, as it has been widely observed experimentally in sands and rocklike materials. If we fix a failure locus, i.e. locus of the peaks, the softening process can be described by taking $D = 0$, as in Cam Clay. Therefore, hardening-softening transition can be accounted for in this way. Cam Clay allowed only for elastic-softening or elastic-hardening behaviour, alternatively. This result is valid whatever the assumption on the shape of plastic potentials and yield loci may be and it is not restricted to the single Cam Clay model.

To illustrate this result, consider a constant p test. Borrowing again the expression of the yield locus and of the plastic potential from Cam Clay, we can express the strain increments as

$$\dot{\varepsilon}^p = \frac{\lambda - k}{M(M-\eta+D)p}, \tag{5.4}$$

$$\dot{v}^p = (M-\eta)\,\dot{\varepsilon}^p. \tag{5.5}$$

No elastic increment of volume can occur since $\dot{p} = 0$. For the sake of simplicity assume again $\dot{\varepsilon}^e = 0$.

Integrating Eqs. (5.4), (5.5) we have:

$$\frac{\eta}{M} = \left(1+\frac{D}{M}\right)\left(1 - e^{-\frac{M\varepsilon}{k}\frac{1}{\frac{\lambda}{k}-1}}\right), \tag{5.6}$$

$$\frac{Mv}{k} = \left(\frac{\lambda}{k}-1\right)\eta - D\frac{M\varepsilon}{k}. \tag{5.7}$$

Equations (5.6), (5.7) are plotted in Figs. 5, 6, 7 for different values of D/M. As in the undrained test, a low value of D/M can model loose sand while a high value of D/M

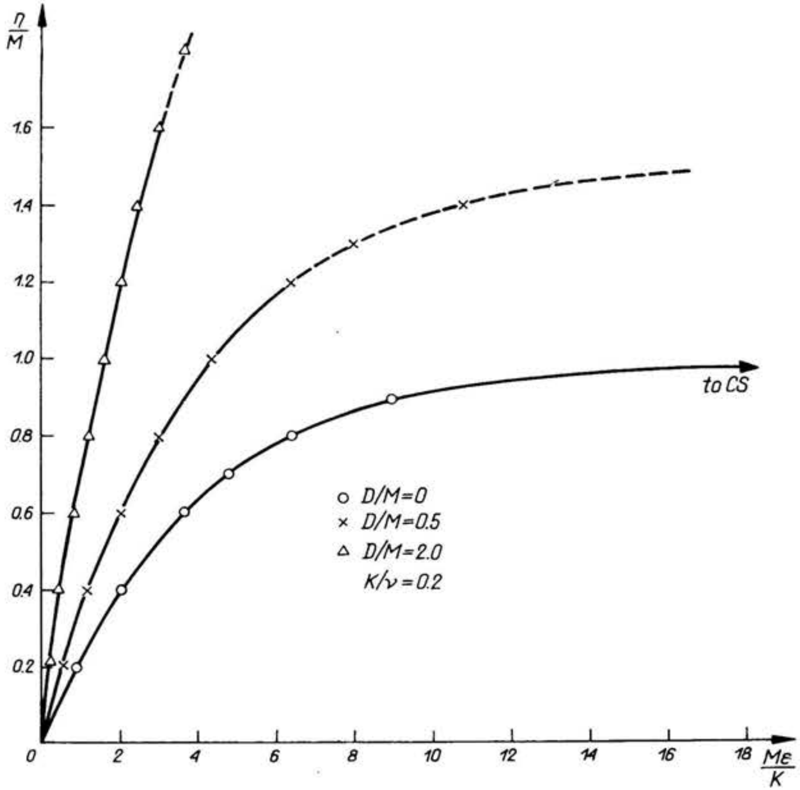

FIG. 5. Constant p test — stress-strain law.

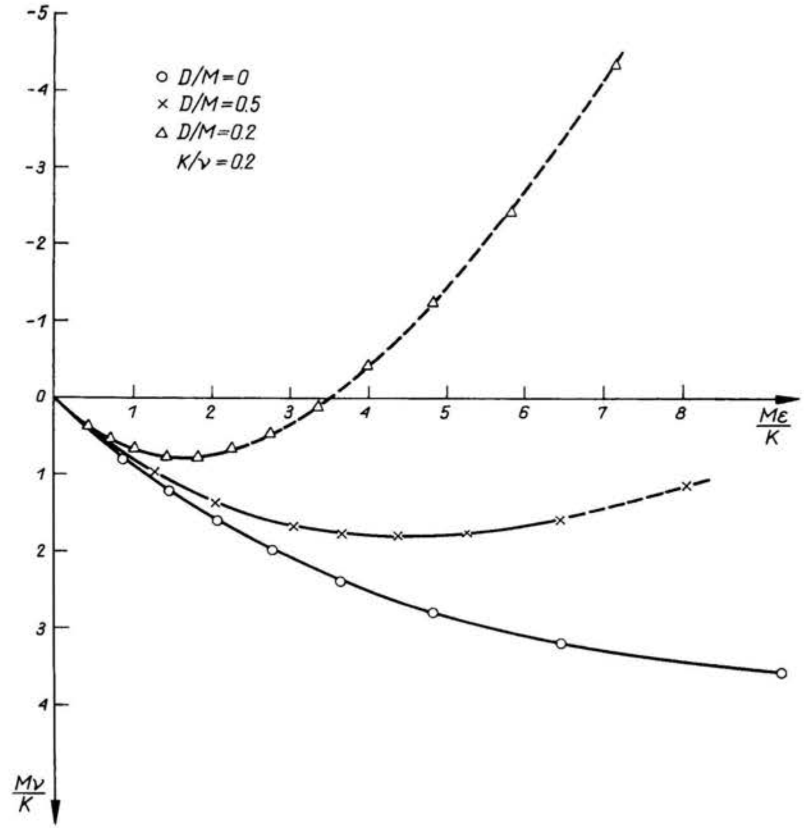

FIG. 6. Constant p test — volumetric versus shear strain.

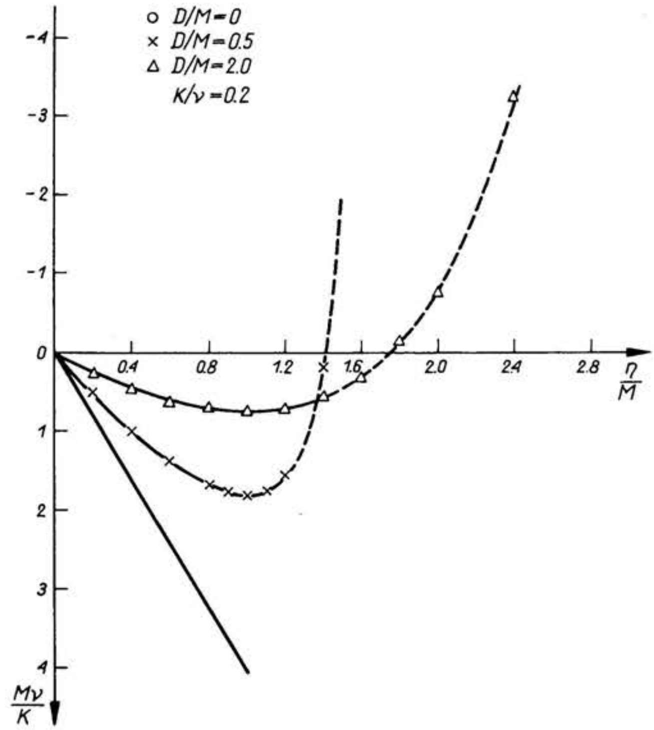

FIG. 7. Constant p test volumetric strain versus stress ratio.

does it for dense sand. A disagreement with experimental findings can be found in Fig. 6. In fact, we see that the curve does not exhibit in its initial part the change in curvature, characteristic in constant p test. This is probably due to the fact that the true yield locus has a vertical tangent at $q = 0$ and only for larger values of η it assumes a shape similar to Cam Clay. The observed disagreement could be removed by correcting the yield locus in this region. It must be emphasized that these equations are valid only during the hardening process.

In the real material at a certain level of the stress ratio depending on the density of the sand the failure locus is attained and a softening process begins. For this reason the lines presented in Figs. 5, 6, 7 are dotted for large values of the stress ratio comparatively to the density. It is questionable whether a p constant test could be performed also during the softening phase and we prefer to leave it aside. Anyway, in principle, Eq. (5.3) should be employed taking $D = 0$.

To compare qualitatively the results obtained in the undrained and p constant tests presented here with experimental data see for example TATSUOKA and ISHIHARA (1973).

A quantitative comparison with real results is meaningless for sands. Indeed the plastic potential used by Schofield and Wroth is not adequate. The Rowe's stress dilatancy relation offers a better interpretation of actual test data. Moreover the assumption of normality embodied in Cam Clay is not correct for sand. An expression for the yield locus different from that of the plastic potential must be used. POOROOHASB (1971) and

TATSUOKA and ISHIHARA (1974) give expressions similar to Cam Clay yield locus but different experimental parameters appear in their equations. Probably this is the reason why the modified Cam Clay presented here gives a good qualitative agreement between calculated and observed data.

7. Conclusions

This note presents an incremental approach to the elastoplastic hardening of soils. It has been shown that in general the plastic strain increments depend not only on the volumetric but also on the shear strain previously experienced by the material specimen. It has also been shown that the general expression presented here can be reduced to Cam Clay once the strain hardening function is expressed in terms of volumetric stains instead of volume change but the choice of appropriate parameters leads to a much better interpretation of actual experimental results. An interesting feature of this model is the description of the "hook" of the effective stress path of sand specimens in undrained conditions. Moreover, the hardening-softening transition observed in sands and in many rocklike materials can, in principle, be described in this way. Some more work should be done to describe quantitatively the hardening, softening behaviour of soil employing more realistic plastic potentials and yield loci.

Appendix A

Equations (3.2) and (3.4) deserve probably more attention. They have been derived from a postulated response to an increment of isotropic stress, Eqs. (3.1), (3.3), respectively. The form of the latter equations has been chosen to match experimental data. If we now integrate Eqs. (3.2) and (3.4), we can derive p_u as a function of the plastic strains.

From Eq. (3.2) we have

(A.1) $$C \ln p_u = v^p + F_1(\varepsilon^p)$$

and from Eq. (3.4)

(A.2) $$C \ln p_u = D\varepsilon^p + F_2(v^p).$$

To satisfy both equations the only possible form for F_1 and F_2 is

(A.3) $$F_1(\varepsilon^p) = D\varepsilon^p + \text{const},$$

(A.4) $$F_2(v^p) = v^p + \text{const},$$

therefore

(A.5) $$C \ln p_u = v^p + D\varepsilon^p + \text{const}.$$

If we assume, as in the rest of the paper, that the material is virgin before the beginning of the loading process and that the surrounding isotropic pressure is one atmosphere when we start the test, the constant in r.h.s. of Eq. (A.5) is zero and we finally have

(A.6) $$p_u = p_u(v^p, \varepsilon^p) = e^{\frac{1}{C}(v^p + D\varepsilon^p)}.$$

Equation (A.5) suggests a way to determine experimentally the family of yield loci. By the definition of p_u, it uniquely determines a yield locus. Thus, on a fixed yield locus

$$v^p + D\varepsilon^p = \text{const}. \tag{A.7}$$

Therefore, suppose we perform a series of tests and then we draw in a $p \div q$ plane the contours for which Eq. (A.7) holds. If we do it for several values of the constant, we get a family of yield loci. Finally, a family of fitting curves approximating the aforementioned contours allows to express analytically the yield function $f(p, q, p_u)$.

Equation (A.7) specializes to

$$v^p = \text{const} \tag{A.8}$$

if $D = 0$. This is a well-known property of the Cam Clay model, following which any yield locus is associated to a definite plastic volumetric strain. In the model presented here, it is not a definite plastic deformation that is associated to a yield locus, but, instead, a linear combination of plastic strains.

References

1. D. C. DRUCKER, W. PRAGER, *Soil mechanics and plastic analysis or limit design*, Quart. of App. Math., **10**, 157–165, 1952.
2. R. L. KONDNER, *Hyperbolic stress-strain response: cohesive soil*, Proc. ASCE J. Soil Mech. Found. Engin., **89**, 115, 1963.
3. R. L. KONDNER, J. S. ZELASKO, *A hyperbolic stress strain formulation for sands*, Proc. 2nd Pan Am. Conf. Soil Mech. Found Eng. Brazil, **1**, 289, 324, 1963.
4. P. V. LADE, D. M. DUNCAN, *Elastoplastic stress strain theory for cohesionless soil*, J. of Geot. Eng. Div. ASCE, **101**, 1037–1053, 1975.
5. D. NAMY, *An investigation of certain aspects of stress strain relationships for clay soils*, Ph. D. Thesis Cornell University, 1970.
6. H. B. POOROOSHASB, *Deformation of sand in triaxial compression*, IV Asian Conf. on Soil Mech. and Found Bangkok, 63–66, 1971.
7. L. PRANDTL, *Über die Harte Plastisher Korper*, Nach. Kgl. Ges. Wiss Gottingen Math. Phys. Klasse 1930.
8. P. W. ROWE, *Theoretical meaning and observed values of deformation parameters for soil*, in: Stress Strain Behaviour of Soils Foulis, 143–194, 1972.
9. A. N. SCHOFIELD, *The development of lateral force during the displacement of sand by the vertical face of a rotating model foundation*, Ph. D. Thesis Cambridge Univ., 1959.
10. A. N. SCHOFIELD, C. P. WROTH, *Critical state soil mechanics*, Mc Graw-Hill, 1968.
11. V. V. SOKOLOVSKII, *Statics of granular media*, Pergamon 1965.
12. F. TATSUOKA, K. ISHIHARA, *Stress path and dilatancy performance of a sand*,Int. Conf. on Soil Mechanics and Foundation Eng., Moscow 1/04 419–424, 1971.
13. F. TATSUOKA, K. ISHIHARA, *Yielding of sand in triaxial compression*, Soils and Foundations, **14**, 2, 63–76, 1974.
14. K. TERZAGHI, *Theoretical soil mechanics*, Wiley 1943.

TECHNICAL UNIVERSITY OF MILANO,
MILANO, ITALY.

Received October 19, 1976.