3.

ON CERTAIN DEFINITE INTEGRALS.

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In the first place, we shall consider the integral

$$V = \iint \dots (\mathfrak{n} \text{ times}) \frac{dx\, dy \dots}{\{(a-x)^2 + (b-y)^2 \dots\}^{\frac{1}{2}\mathfrak{n}-1}},$$

the integration extending to all real values of the variables, subject to the condition

$$\frac{x^2}{h^2} + \frac{y^2}{h_,^2} + \dots < \text{ or } = 1,$$

and the constants a, b, &c. satisfying the condition

$$\frac{a^2}{h^2} + \frac{b^2}{h_,^2} \dots > 1.$$

We have

$$\frac{dV}{da} = -(\mathfrak{n}-2) \iint \dots (\mathfrak{n} \text{ times}) \frac{(a-x)\, dx\, dy \dots}{\{(a-x)^2 + (b-y)^2 \dots\}^{\frac{1}{2}\mathfrak{n}}},$$

$$= -(\mathfrak{n}-2) \frac{2hh_, \dots \pi^{\frac{1}{2}\mathfrak{n}} a}{\sqrt{(\xi + h^2)} \cdot \Gamma(\frac{1}{2}\mathfrak{n})} \int_0^1 \frac{x^{\mathfrak{n}-1}\, dx}{\sqrt{[\{\xi + h^2 + (h_,^2 - h^2)\, x^2\}\{\xi + h^2 + (h_{,,}^2 - h^2)\, x^2\} \dots]}},$$

ξ being determined by the equation

$$\frac{a^2}{\xi + h^2} + \frac{b^2}{\xi + h_,^2} \dots = 1,$$

by a formula [see p. 12] in a paper, [2], "On the Properties of a Certain Symbolical Expression," in the preceding No. of this Journal: ξ having been substituted for the η^2 of the formula.

Let the variable x, on the second side of the equation, be replaced by ϕ, where

$$x^2 = \frac{\xi + h^2}{\xi + h^2 + \phi};$$

we have without difficulty

$$\frac{dV}{da} = -(\mathfrak{n}-2)\frac{hh_, \ldots \pi^{\frac{1}{2}\mathfrak{n}} a}{\Gamma(\frac{1}{2}\mathfrak{n})}\int_0^\infty \frac{d\phi}{(\xi + h^2 + \phi)\sqrt{\Phi}},$$

where

$$\Phi = (\xi + h^2 + \phi)(\xi + h_,^2 + \phi)\ldots$$

and similarly

$$\frac{dV}{db} = -(\mathfrak{n}-2)\frac{hh_, \ldots \pi^{\frac{1}{2}\mathfrak{n}} b}{\Gamma(\frac{1}{2}\mathfrak{n})}\int_0^\infty \frac{d\phi}{(\xi + h_,^2 + \phi)\sqrt{\Phi}},$$

&c. ......

From these values it is easy to verify the equation

$$V = \frac{(\mathfrak{n}-2)hh_, \ldots \pi^{\frac{1}{2}\mathfrak{n}}}{2\Gamma(\frac{1}{2}\mathfrak{n})}\int_0^\infty \left(1 - \frac{a^2}{\xi + h^2 + \phi} - \frac{b^2}{\xi + h_,^2 + \phi}\ldots\right)\frac{d\phi}{\sqrt{\Phi}}.$$

For this evidently verifies the above values of $\frac{dV}{da}$, $\frac{dV}{db}$, &c. if only the term $\frac{dV}{d\xi}d\xi$ vanishes; and we have

$$\frac{dV}{d\xi} = \frac{(\mathfrak{n}-2)hh_, \ldots \pi^{\frac{1}{2}\mathfrak{n}}}{2\Gamma(\frac{1}{2}\mathfrak{n})}\int_0^\infty d\phi \cdot \frac{d}{d\xi}\left(1 - \frac{a^2}{\xi + h^2 + \phi}\ldots\right)\frac{1}{\sqrt{\Phi}};$$

or, observing that

$$\frac{d}{d\xi}\left(1 - \frac{a^2}{\xi + h^2 + \phi} - \ldots\right)\frac{1}{\sqrt{(\Phi)}} = \frac{d}{d\phi}\left(1 - \frac{a^2}{\xi + h^2 + \phi}\ldots\right)\frac{1}{\sqrt{(\Phi)}},$$

and taking the integral from 0 to ∞,

$$\frac{dV}{d\xi} = -\frac{(\mathfrak{n}-2)hh_, \ldots \pi^{\frac{1}{2}\mathfrak{n}}}{2\Gamma(\frac{1}{2}\mathfrak{n})}\left(1 - \frac{a^2}{\xi + h^2} - \frac{b^2}{\xi + h_,^2}\ldots\right)\frac{1}{\sqrt{\{(\xi + h^2)(\xi + h_,^2)\ldots\}}}, = 0,$$

in virtue of the equation which determines ξ.

No constant has been added to the value of V, since the two sides of the equation vanish as they should do for a, $b \ldots$ infinite, for which values ξ is also infinite and the quantity

$$\left(1 - \frac{a^2}{\xi + h^2 + \phi}\ldots\right)\frac{1}{\sqrt{(\Phi)}},$$

which is always less than $\frac{1}{\sqrt{(\Phi)}}$, vanishes.

Hence, restoring the values of V and Φ,

$$\iint\ldots(\mathfrak{n}\text{ times})\frac{dx\,dy\ldots}{\{(a-x)^2 + (b-y)^2\ldots\}^{\frac{1}{2}\mathfrak{n}-1}}$$

$$= \frac{(\mathfrak{n}-2)hh_, \ldots \pi^{\frac{1}{2}\mathfrak{n}}}{2\Gamma(\frac{1}{2}\mathfrak{n})}\int_0^\infty \left(1 - \frac{a^2}{\xi + h^2 + \phi} - \frac{b^2}{\xi + h_,^2 + \phi}\ldots\right)\frac{d\phi}{\sqrt{\{(\xi + h^2 + \phi)(\xi + h_,^2 + \phi)\ldots\}}}$$

the limits of the first side of the equation, and the condition to be satisfied by a, b, &c., also the equation for the determination of ξ, being as above.

The integral

$$V' = \iint \dots (\mathfrak{n} \text{ times}) \frac{dx\, dy \dots}{\{(a-x)^2 + (b-y)^2 \dots\}^{\frac{1}{2}\mathfrak{n}}},$$

between the same limits, and with the same condition to be satisfied by the constants, has been obtained [see p. 11] in the paper already quoted. Writing ξ instead of η^2, and $x^2 = \dfrac{\xi}{\xi + \phi}$, we have

$$V' = \frac{hh_, \dots \pi^{\frac{1}{2}\mathfrak{n}}}{\Gamma(\frac{1}{2}\mathfrak{n})} \int_0^\infty \frac{d\phi}{(\xi+\phi)\sqrt{\{(\xi+h^2+\phi)(\xi+h_,^2+\phi)\dots\}}},$$

where $$\frac{a^2}{\xi+h^2} + \frac{b^2}{\xi^2 + h_,^2} \dots = 1.$$

Let $\nabla = \dfrac{d^2}{da^2} + \dfrac{d^2}{db^2} + \dots$ Then by the assistance of a formula,

$$\nabla^q \frac{1}{(a^2+b^2\dots)^i} = 2i(2i+2)\dots(2i+2q-2)(2i+2-\mathfrak{n})\dots(2i+2q-\mathfrak{n}) \cdot \frac{1}{(a^2+b^2\dots)^{i+q}}$$

given in the same paper [see p. 6], in which it is obvious that $a, b \dots$ may be changed into $a-x$, $b-y$, &c. ...; also putting $i = \frac{1}{2}\mathfrak{n}$; we have

$$\iint \dots (\mathfrak{n} \text{ times}) \frac{dx\, dy \dots}{\{(a-x)^2 + \dots\}^{\frac{1}{2}\mathfrak{n}+q}} = \frac{hh_, \dots\dots \pi^{\frac{1}{2}\mathfrak{n}}}{2^{2q}.1.2\dots q.\Gamma(\frac{1}{2}\mathfrak{n}+q)} \int_0^\infty d\phi \,.\, \nabla^q \frac{1}{(\xi+\phi)\sqrt{\{(\xi+h^2+\phi)\dots\}}}.$$

Now in general, if $\chi\xi$ be any function of ξ,

$$\nabla\chi\xi = \chi'\xi\left(\frac{d^2\xi}{da^2} + \frac{d^2\xi}{db^2}\dots\right) + \chi''\xi\left\{\left(\frac{d\xi}{da}\right)^2 + \left(\frac{d\xi}{db}\right)^2\dots\right\} = \chi'\xi\,\Sigma\left(\frac{d^2\xi}{da^2}\right) + \chi''\xi\,\Sigma\left(\frac{d\xi}{da}\right)^2, \text{ suppose.}$$

But from the equation $$\Sigma\frac{a^2}{(\xi+h^2)} = 1,$$

we obtain $$\frac{2a}{\xi+h^2} - \left\{\Sigma\frac{a^2}{(\xi+h^2)^2}\right\}\frac{d\xi}{da} = 0,$$

whence $$\Sigma\left(\frac{d\xi}{da}\right)^2 = \frac{4}{\Sigma\dfrac{a^2}{(\xi+h^2)^2}}.$$

Also $$\frac{2}{\xi+h^2} - 4\frac{a}{(\xi+h^2)^2}\frac{d\xi}{da} + 2\left\{\Sigma\frac{a^2}{(\xi+h^2)^3}\right\}\left(\frac{d\xi}{da}\right)^2 - \left\{\Sigma\frac{a^2}{(\xi+h^2)^2}\right\}\frac{d^2\xi}{da^2} = 0;$$

whence taking the sum Σ, and observing that

$$-4\Sigma\frac{a}{(\xi+h^2)^2}\frac{d\xi}{da} = -8\frac{\Sigma\dfrac{a^2}{(\xi+h^2)^3}}{\Sigma\dfrac{a^2}{(\xi+h^2)^2}} = -2\Sigma\frac{a^2}{(\xi+h^2)^3}\cdot\Sigma\left(\frac{d\xi}{da}\right)^2,$$

$$2\Sigma\frac{1}{\xi+h^2} - \left\{\Sigma\frac{a^2}{(\xi+h^2)^2}\right\}\Sigma\left(\frac{d^2\xi}{da^2}\right) = 0;$$

we find
$$\Sigma\left(\frac{d^2\xi}{da^2}\right)=\frac{2\Sigma\dfrac{1}{\xi+h^2}}{\Sigma\dfrac{a^2}{(\xi+h^2)^2}};$$

and we hence obtain
$$\nabla\chi\xi=\frac{2\chi'\xi\,\Sigma\dfrac{1}{\xi+h^2}+4\,\chi''\xi}{\Sigma\dfrac{a^2}{(\xi+h^2)^2}}.$$

Hence the function
$$\int_0^\infty d\phi\,.\,\nabla\frac{1}{(\xi+\phi)\sqrt{\{(\xi+h^2+\phi)\ldots\}}}$$
(observing that differentiation with respect to ξ is the same as differentiation with respect to ϕ) becomes integrable, and taking the integral between the proper limits, its value is
$$-\frac{2\chi_0\xi\,\Sigma\dfrac{1}{\xi+h^2}+4\chi_0'\xi}{\Sigma\dfrac{a^2}{(\xi+h^2)^2}};$$

where
$$\chi_0\xi=\frac{1}{\xi\sqrt{\{(\xi+h^2)(\xi+h_{,}^2)\ldots\}}}.$$

We have immediately
$$\frac{\chi_0'\xi}{\chi_0\xi}=-\tfrac{1}{2}\left(\frac{2}{\xi}+\Sigma\frac{1}{\xi+h^2}\right);$$

or
$$2\chi_0\xi\,\Sigma\left(\frac{1}{\xi+h^2}\right)+4\chi_0'\xi=-4\,\frac{\chi_0\xi}{\xi};$$

whence
$$\int_0^\infty d\phi\,.\,\nabla\frac{1}{(\xi+\phi)\sqrt{\{(\xi+h^2+\phi)\ldots\}}}=\frac{4}{\xi^2\sqrt{\{(\xi+h^2)(\xi+h_{,}^2)\ldots\}}\left\{\dfrac{a^2}{(\xi+h^2)^2}+\dfrac{b^2}{(\xi+h_{,}^2)^2}+\ldots\right\}}.$$

Hence restoring the value of ∇, and of the first side of the equation,
$$\iint\ldots(\mathfrak{n}\text{ times})\frac{dx\,dy\ldots}{\{(a-x)^2+(b-y)^2\ldots\}^{\frac{1}{2}\mathfrak{n}+q}}$$
$$=\frac{hh_{,}\ldots\pi^{\frac{1}{2}\mathfrak{n}}}{2^{2q-2}.1.2\ldots q.\Gamma(\frac{1}{2}\mathfrak{n}+q)}\left(\frac{d^2}{da^2}+\frac{d^2}{db^2}\ldots\right)^{q-1}\frac{1}{\xi^2\sqrt{\{(\xi+h^2)(\xi+h_{,}^2)\ldots\}}\left\{\dfrac{a^2}{(\xi+h^2)^2}+\dfrac{b^2}{(\xi+h_{,}^2)^2}+\ldots\right\}},$$

with the condition
$$\frac{a^2}{\xi+h^2}+\frac{b^2}{\xi+h_{,}^2}\ldots=1;$$

from which equation the differential coefficients of ξ, which enter into the preceding result, are to be determined.

In general if u be any function of ξ, a, b ...

$$\left(\frac{d^2}{da^2}+\frac{d^2}{db^2}\dots\right)u=\frac{4\dfrac{d^2u}{d\xi^2}+2\dfrac{du}{d\xi}\Sigma\dfrac{1}{\xi+h^2}+4\dfrac{d^2u}{d\xi da}\Sigma\dfrac{a}{\xi+h^2}}{\Sigma\dfrac{a^2}{(\xi+h^2)^2}}+\Sigma\frac{d^2u}{da^2},$$

from which the values of the second side for $q=1$, $q=2$, &c. may be successively calculated.

The performance of the operation $\left(\frac{d}{da}\right)^p\left(\frac{d}{db}\right)^q\left(\frac{d}{dc}\right)^r$..., upon the integral V', leads in like manner to a very great number of integrals, all of them expressible algebraically, for a single differentiation renders the integration with respect to ϕ possible. But this is a subject which need not be further considered at present.

We shall consider, lastly, the definite integral

$$U=\iint\dots(\mathfrak{n}\text{ times})\frac{(a-x)f\left(\dfrac{x^2}{h^2}+\dfrac{y^2}{h_{\prime}^{\,2}}+\dots\right)dxdy\dots}{\{(a-x)^2+(b-y)^2\dots\}^{\frac{1}{2}\mathfrak{n}}},$$

limits, &c. as before. This is readily deduced from the less general one

$$\iint\dots(\mathfrak{n}\text{ times})\frac{(a-x)\,dxdy\dots}{\{(a-x)^2+(b-y)^2\dots\}^{\frac{1}{2}\mathfrak{n}}}.$$

For representing this quantity by $F(h,\,h_{\prime}\dots)$, it may be seen that

$$U=\int_0^1 f(m^2)\frac{d}{dm}F(mh,\;mh_{\prime}\dots)\,dm;$$

but in the value of $F(h,\,h_{\prime}\dots)$, changing h, $h_{\prime}$... into mh, $mh_{\prime}$... also writing $m^2\phi$ instead of ϕ, and $m^2\xi'$ for ξ, we have

$$F(mh,\;mh_{\prime}\dots)=\frac{hh_{\prime}\dots\pi^{\frac{1}{2}\mathfrak{n}}}{\Gamma(\frac{1}{2}\mathfrak{n})}a\int_0^\infty\frac{d\phi}{(\xi'+h^2+\phi)\sqrt{(\Phi')}};$$

where $$\Phi'=(\xi'+h^2+\phi)(\xi'+h_{\prime}^{\,2}+\phi)\dots$$

and $$\frac{a^2}{\xi'+h^2}+\frac{b^2}{\xi'+h_{\prime}^{\,2}}+\dots=m^2.$$

Hence $$\frac{d}{dm}F(mh,\;mh_{\prime}\dots)=\frac{d\xi'}{dm}\frac{d}{d\xi'}F(mh,\;mh_{\prime}\dots),$$

$$=\frac{hh_{\prime}\dots\pi^{\frac{1}{2}\mathfrak{n}}}{\Gamma(\frac{1}{2}\mathfrak{n})}a\frac{d\xi'}{dm}\int_0^\infty d\phi\frac{d}{d\xi'}\frac{1}{(\xi'+h^2+\phi)\sqrt{(\Phi')}},$$

or, observing that $\frac{d}{d\xi'}$ is equivalent to $\frac{d}{d\phi}$, and effecting the integration between the proper limits,

$$\frac{d}{dm}F(mh,\;mh_{\prime}\dots)=-\frac{hh_{\prime}\dots\pi^{\frac{1}{2}\mathfrak{n}}}{\Gamma(\frac{1}{2}\mathfrak{n})}a\frac{1}{(\xi'+h^2)\sqrt{\{(\xi'+h^2)(\xi'+h_{\prime}^{\,2})\dots\}}}.$$

Substituting this value, also $f\left\{\frac{a^2}{\xi'+h^2}+\frac{b^2}{\xi'+h_{,}^2}+\dots\right\}$ for $f(m^2)$, in the value of U, and observing that $m=0$ gives $\xi'=\infty$, $m=1$ gives $\xi'=\xi$, where ξ is a quantity determined as before by the equation

$$\frac{a^2}{\xi+h^2}+\frac{b^2}{\xi+h_{,}^2}+\dots=1,$$

we have $U=-\dfrac{hh_{,}\dots\pi^{\frac{1}{2}\mathfrak{n}}a}{\Gamma(\frac{1}{2}\mathfrak{n})}\displaystyle\int_{\infty}^{\xi}\frac{f\left\{\frac{a^2}{\xi'+h^2}+\dots\right\}d\xi'}{(\xi'+h^2)\sqrt{\{(\xi'+h^2)(\xi'+h_{,}^2)\dots\}}}$,

or writing $\phi+\xi$ for ξ', $d\xi'=d\phi$, the limits of ϕ are 0, ∞; or, inverting the limits and omitting the negative sign,

$$U=\frac{hh_{,}\dots\pi^{\frac{1}{2}\mathfrak{n}}a}{\Gamma(\frac{1}{2}\mathfrak{n})}\int_0^{\infty}\frac{f\left\{\frac{a^2}{\xi+h^2+\phi}+\frac{b^2}{\xi+h_{,}^2+\phi}+\dots\right\}d\phi}{(\xi+h^2+\phi)\sqrt{\{(\xi+h^2+\phi)(\xi+h_{,}^2+\phi)\dots\}}};$$

which, in the particular case of $\mathfrak{n}=3$, may easily be made to coincide with known results. The analogous integral

$$\iint\dots(\mathfrak{n}\text{ times})\frac{f\left\{\frac{x^2}{h^2}+\frac{y^2}{h_{,}^2}+\dots\right\}dx\,dy\dots}{\{(a-x)^2+(b-y)^2\dots\}^{\frac{1}{2}\mathfrak{n}}}$$

is apparently not reducible to a single integral.