21.

ON JACOBI'S ELLIPTIC FUNCTIONS, IN REPLY TO THE REV. B. BRONWIN; AND ON QUATERNIONS.

[From the *Philosophical Magazine*, vol. XXVI. (1845), pp. 208, 211.]

The first part of this Paper is omitted, see [17]: only the Postscript on Quaternions, pp. 210, 211, is printed.

It is possible to form an analogous theory with seven imaginary roots of (-1) (? with $\nu = 2^n - 1$ roots when ν is a prime number). Thus if these be $i_1, i_2, i_3, i_4, i_5, i_6, i_7$, which group together according to the types

123, 145, 624, 653, 725, 734, 176,

i.e. the type **123** denotes the system of equations

$$\begin{array}{lll} i_1 i_2 = i_3, & i_2 i_3 = i_1, & i_3 i_1 = i_2, \\ i_2 i_1 = - i_3, & i_3 i_2 = - i_1, & i_1 i_3 = - i_2, \end{array}$$

&c. We have the following expression for the product of two factors:

$$\begin{array}{l} (X_0 + X_1 i_1 + \dots X_7 i_7)(X'_0 + X'_1 i_1 + \dots X'_7 i_7) \\ \quad = X_0 X'_0 - X_1 X'_1 - X_2 X'_2 \dots - X_7 X'_7 \\ \qquad + [\overline{23} + \overline{45} + \overline{76} + (01)]\, i_1 \\ \qquad + [\overline{31} + \overline{46} + \overline{57} + (02)]\, i_2 \\ \qquad + [\overline{12} + \overline{65} + \overline{47} + (03)]\, i_3 \\ \qquad + [\overline{51} + \overline{62} + \overline{47} + (04)]\, i_4 \\ \qquad + [\overline{14} + \overline{36} + \overline{72} + (05)]\, i_5 \\ \qquad + [\overline{24} + \overline{53} + \overline{17} + (06)]\, i_6 \\ \qquad + [\overline{25} + \overline{34} + \overline{61} + (07)]\, i_7 \end{array}$$

where $(01) = X_0 X'_1 + X_1 X'_0 \dots; \ \overline{12} = X_1 X'_2 - X_2 X'_1$ &c.;

and the modulus of this expression is the product of the moduli of the factors. The above system of types requires some care in writing down, and not only with respect to the combinations of the letters, but also to their order; it would be vitiated, e.g. by writing 716 instead of 176. A theorem analogous to that which I gave before, for quaternions, is the following:—If $\Lambda = 1 + \lambda_1 i_1 \dots + \lambda_7 i_7$, $X = x_1 i_1 \dots + x_7 i_7$: it is immediately shown that the possible part of $\Lambda^{-1} X \Lambda$ vanishes, and that the coefficients of $i_1, \dots i_7$ are linear functions of $x_1, \dots x_7$. The modulus of the above expression is evidently the modulus of X; hence "we may determine seven linear functions of $x_1 \dots x_7$, the sum of whose squares is equal to $x_1^2 + \dots + x_7^2$." The number of arbitrary quantities is however only seven, instead of twenty-one, as it should be.