22.

ON ALGEBRAICAL COUPLES.

[From the *Philosophical Magazine*, vol. XXVII. (1845), pp. 38—40.]

It is worth while, in connection with the theory of quaternions and the researches of Mr Graves (*Phil. Mag.* [Vol. XXVI. (1845), pp. 315—320]), to investigate the properties of a couple $ix+jy$ in which i, j are symbols such that

$$\begin{aligned} i^2 &= \alpha i + \beta j, \\ ij &= \alpha' i + \beta' j, \\ ji &= \gamma i + \delta j, \\ j^2 &= \gamma' i + \delta' j. \end{aligned}$$

If
$$\overline{ix+jy}\ \overline{ix_1+jy_1} = iX + jY,$$
then
$$\begin{aligned} X &= \alpha\, xx_1 + \alpha' xy_1 + \gamma x_1 y + \gamma' yy_1, \\ Y &= \beta' xx_1 + \beta' xy_1 + \delta' x_1 y + \delta' yy_1. \end{aligned}$$

Imagine the constants α, β ... so determined that $ix+jy$ may have a modulus of the form $K(x+\lambda y)(x+\mu y)$; there results one of the four following essentially independent systems

A. $$i^2 = \frac{1}{\lambda\mu}(\delta\lambda\mu + \gamma\,\overline{\lambda+\mu})\, i - \frac{\gamma}{\lambda\mu} j,$$

$$ij = ji = \gamma i + \delta j,$$

$$j^2 = -\lambda\mu\delta i + (\gamma + \overline{\lambda+\mu}\,\delta)\, j,$$

$$\left\{\begin{aligned} X + \lambda Y &= \frac{1}{\lambda}(\gamma + \lambda\delta)(x+\lambda y)(x_1 + \lambda y_1), \\ X + \mu Y &= \frac{1}{\mu}(\gamma + \mu\delta)(x+\mu y)(x + \mu y_1). \end{aligned}\right.$$

The couple may be said to have the *two linear moduli,*

$$\frac{1}{\lambda}(\gamma+\lambda\delta)(x+\lambda y),\quad \frac{1}{\mu}(\gamma+\mu\delta)(x+\mu y);$$

as well as the quadratic one,

$$\frac{1}{\lambda\mu}(\gamma+\lambda\delta)(\gamma+\mu\delta)(x+\lambda y)(x+\mu y),$$

the product of these, which is the modulus, and the only modulus in the remaining systems.

B.

$$i^2=-\delta i+\frac{1}{\lambda\mu}(\gamma+\delta\overline{\lambda+\mu})j,$$

$$ij=ji=\gamma i+\delta j,$$

$$j^2=(\overline{\lambda+\mu}\,\gamma+\lambda\mu\delta)\,i-\gamma j,$$

$$\begin{cases} X+\lambda Y=\dfrac{1}{\mu}(\gamma+\lambda\delta)(x+\mu y)(x_1+\mu y_1),\\[2ex] X+\mu Y=\dfrac{1}{\lambda}(\gamma+\mu\delta)(x+\lambda y)(x_1+\lambda y_1).\end{cases}$$

C.

$$i^2=\frac{1}{\lambda\mu}(\delta\lambda\mu+\gamma\,\overline{\lambda+\mu})\,i-\frac{\gamma}{\lambda\mu}j,$$

$$ij=\left(\frac{\mu^2+\mu\lambda+\lambda^2}{\mu\lambda}\gamma+\overline{\mu+\lambda}\,\delta\right)i+\left(-\delta-\frac{\lambda+\mu}{\lambda\mu}\gamma\right)j,$$

$$ji=\quad\gamma i+\delta j,$$

$$j^2=(\overline{\lambda+\mu}\gamma+\lambda\mu\delta)\,i-\gamma j,$$

$$\begin{cases} X+\lambda Y=\dfrac{1}{\lambda}(\gamma+\lambda\delta)(x+\lambda y)(x_1+\mu y_1),\\[2ex] X+\mu Y=\dfrac{1}{\mu}(\gamma+\mu\delta)(x+\mu y)(x_1+\lambda y_1).\end{cases}$$

D.

$$i^2=-\delta i+\frac{1}{\lambda\mu}(\gamma+\delta\,\overline{\lambda+\mu})j,$$

$$ij=(-\gamma-\overline{\lambda+\mu}\delta)\,i+\left(\frac{\lambda+\mu}{\lambda\mu}\gamma+\frac{\lambda^2+\lambda\mu+\mu^2}{\lambda\mu}\delta\right)j,$$

$$ji=\quad\gamma i+\delta j,$$

$$j^2=-\lambda\mu\delta i+(\gamma+\overline{\lambda+\mu}\delta)j,$$

$$\begin{cases} X+\lambda Y=\dfrac{1}{\mu}(\gamma+\lambda\delta)(x+\mu y)(x_1+\lambda y_1),\\[2ex] X+\mu Y=\dfrac{1}{\lambda}(\gamma+\mu\delta)(x+\lambda y)(x_1+\mu y_1).\end{cases}$$

The formulæ are much simpler and not essentially less general, if $\mu = -\lambda$. They thus become

A′.
$$\begin{aligned} i^2 &= \delta i + \frac{\gamma}{\lambda^2} j, \\ ij = ji &= \gamma i + \delta j, \\ j^2 &= \lambda^2 \delta i + \gamma j, \\ X \pm \lambda Y &= \pm \frac{1}{\lambda} (\gamma \pm \lambda\delta)(x \pm \lambda y)(x_1 \pm \lambda y_1). \end{aligned}$$

(Two linear moduli.)

B′.
$$\begin{aligned} i^2 &= -\delta i - \frac{\gamma}{\lambda^2} j, \\ ij = ji &= \gamma i + \delta j, \\ j^2 &= -\lambda^2 \delta i - \gamma j, \\ X \pm \lambda Y &= \mp \frac{1}{\lambda} (\gamma \pm \lambda\delta)(x \mp \lambda y)(x_1 \mp \lambda y_1). \end{aligned}$$

C′.
$$\begin{aligned} i^2 &= -\delta i + \frac{\gamma}{\lambda^2} j, \\ ij &= -\gamma i - \delta j, \\ ji &= \gamma i + \delta j, \\ j^2 &= -\lambda^2 \delta i - \gamma j, \\ X \pm \lambda Y &= \pm \frac{1}{\lambda} (\gamma \pm \lambda\delta)(x \pm \lambda y)(x_1 \mp \lambda y_1). \end{aligned}$$

D′.
$$\begin{aligned} i^2 &= -\delta i - \frac{\gamma}{\lambda^2} j, \\ ij &= -\gamma i - \delta j, \\ ji &= \gamma i + \delta j, \\ j^2 &= \lambda^2 \delta i + \gamma j, \\ X \pm \lambda Y &= \mp \frac{1}{\lambda} (\gamma \pm \lambda\delta)(x \mp \lambda y)(x_1 \pm \lambda y). \end{aligned}$$

There is a system more general than (A.) having a *single* linear modulus $q(\theta X + Y)$: this is

E.
$$\begin{aligned} i^2 &= \alpha (i - \theta j) + \theta^2 q j, \\ ij &= \alpha' (i - \theta j) + \theta q j, \\ ji &= \gamma (i - \theta j) + \theta q j, \\ j^2 &= \gamma' (i - \theta j) + q j, \\ \theta X + Y &= q (\theta x + y)(\theta x_1 + y_1); \end{aligned}$$

or, without real loss of generality,

E′.
$$i^2 = \alpha i,$$
$$ij = \alpha' i,$$
$$ji = \gamma i,$$
$$j^2 = \gamma' i + qj,$$
$$Y = qyy_1.$$

To complete the theory of this system, one may add the identical equation

$$X + \frac{1}{\theta - qM} Y = \frac{q(\theta^2 - M\alpha)}{\theta - qM} \left(x + \frac{\alpha' - \theta\gamma}{\alpha - \theta\alpha'} y\right) \left(x_1 + \frac{\alpha' - \theta\gamma'}{\alpha - \theta\gamma} y_1\right),$$

where
$$M = \frac{\theta(\gamma - \theta\gamma') - (\alpha - \theta\alpha')}{\alpha'\gamma - \alpha\gamma'}.$$

By determining the constants, so that $\dfrac{1}{\theta - qM} = \dfrac{\alpha' - \theta\gamma}{\alpha - \theta\alpha'} = \dfrac{\alpha' - \theta\gamma'}{\alpha - \theta\gamma}$, the system would reduce itself to the form A.