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ON THE GEOMETRICAL REPRESENTATION OF THE MOTION OF A SOLID BODY.

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Let P, Q, R, ... be consecutive generating lines of a skew surface, and on these take points p', p; q', q; r', r ...... such that pq', qr' ...... are the shortest distances between P and Q, Q and R, &c. Then for the generating line P, the ratio of the inclination of the lines P, Q to the distance pq' is said to be "the torsion," the angle $q'pq$ is said to be the deviation, and the ratio of the inclination of the planes Qpq' and Qqr' to the inclination of P and Q is said to be the "skew curvature." And similarly for any other generating line; so that the torsion and deviation depend on the position of the consecutive line, and the skew curvature on the position of the two consecutive lines. The curve pqr ...... is said to be the minimum distance curve [or curve of striction]. {When the skew surface degenerates into a developable surface, the torsion is infinite, the deviation a right angle, the skew curvature proportional to the curvature of the principal section, i.e. it is the distance of a point from the edge of regression, multiplied into the reciprocal of the radius of curvature, a product which is evidently constant along a generating line. Also the curve of minimum distance becomes the edge of regression.} A skew surface, considered independently of its position in space, is determined when for each generating line we know the torsion, deviation, and skew curvature. For, assuming arbitrarily the line P and the point p, also the plane in which pq' lies, the position of Q is completely determined from the given torsion and deviation; and then Q being known, the position of R is completely determined from the skew curvature for P, and the torsion and deviation for Q; and similarly the consecutive generating lines are to be determined.

Two skew surfaces are said to be "deformations" of each other, when for corresponding generating lines the torsion is always the same. Thus a surface will be deformed if considering the elements between the successive generating lines P, Q ...... as rigid, these elements be made to revolve round the successive generating lines P, Q ...... and to slide along them. {They are "transformations", when not only the torsions but also the deviations are equal at corresponding generating lines: thus, if the sliding of the elements along P, Q ...... be omitted, the new surface will be, not a deformation, but a transformation of the other.} No two skew surfaces can be made to roll and slide one upon the other, so that their successive generating lines coincide, unless one of them is a deformation of the other: and when this is the case, the rolling and sliding motions are *completely determined.* In fact the angular velocity of the generating line is the angular velocity round this line, into the difference of the skew curvatures of the two surfaces; the velocity of translation of the generating line in its own direction is to the angular velocity of the generating line, as the difference of the deviations is to the torsion. {This includes also the case in which one surface is a transformation of the other, where the motion is evidently a rolling one.} A skew surface moving in this manner upon another of which it is the deformation, may be said to "glide" upon it. We may now state the kinematical theorem:

"Any motion whatever of a solid body in space may be represented as the 'gliding' motion of one skew surface upon another fixed in space, and of which it is the deformation."

a theorem which is to be considered as the generalization of the well-known one—

"Any motion of a solid body round a fixed point may be represented as the rolling motion of a conical surface upon a second conical surface fixed in space."

and of the supplementary theorem—

"The angular velocity round the line of contact (the instantaneous axis) is to the angular velocity of this line as the difference of curvatures of the two cones at any point in the same line, to the reciprocal of the distance of the point from the vertex."

The analytical demonstration of this last theorem is rather interesting: it depends on the following formulæ. Forming two determinants, the first with the angular velocities round three axes fixed in space, and the first and second derived coefficients of these velocities with respect to the time; the other in the same way with the angular velocities round axes fixed in the body; the difference of these determinants is equal to the fourth power of the angular velocity into the square of the angular velocity of the instantaneous axis.

To show this, let p, q, r be the angular velocities round the axes fixed in the body; u, v, w those round axes fixed in space; ω the angular velocity round the instantaneous axis; ∇, Ω the two determinants: the theorem comes to

$$\nabla - \Omega = M,$$

where

$$M = \omega^2 (p'^2 + q'^2 + r'^2 - \omega'^2), \text{ or } \omega^2 (u'^2 + v'^2 + w'^2 - \omega'^2).$$

Here

$$\begin{aligned} u &= \alpha p + \beta q + \gamma r, \\ v &= \alpha' p + \beta' q + \gamma' r, \\ w &= \alpha'' p + \beta'' q + \gamma'' r; \end{aligned}$$

whence

$$u' = \alpha p' + \beta q' + \gamma r',$$
$$v' = \alpha' p' + \beta' q' + \gamma' r',$$
$$w' = \alpha'' p' + \beta'' q' + \gamma'' r',$$

(the remaining terms vanishing as is well known); and therefore

$$vw' - v'w = \alpha\ (qr' - q'r) + \beta\ (rp' - r'p) + \gamma\ (pq' - p'q),$$
$$wu' - w'u = \alpha'\ (qr' - q'r) + \beta'\ (rp' - r'p) + \gamma'\ (pq' - p'q),$$
$$uv' - u'v = \alpha''\ (qr' - q'r) + \beta''\ (rp' - r'p) + \gamma''\ (pq' - p'q).$$

Hence

$$vw'' - v''w = \alpha\ (qr'' - q''r) + \beta\ (rp'' - r''p) + \gamma\ (pq'' - p''q) + u'\omega^2 - u\omega\omega',$$
$$wu'' - w''u = \alpha'\ (qr'' - q''r) + \beta'\ (rp'' - r''p) + \gamma'\ (pq'' - p''q) + v'\omega^2 - v\omega\omega',$$
$$uv'' - u''v = \alpha''\ (qr'' - q''r) + \beta''\ (rp'' - r''p) + \gamma''\ (pq'' - p''q) + w'\omega^2 - w\omega\omega',$$

and multiplying these by u', v', w', and adding, the required equation is immediately obtained.

In fact, if r be the distance of a point in the instantaneous axis from the vertex, and ρ, σ the radii of curvature of the two cones at that point, then

$$\frac{r}{\rho} = \frac{\omega^3}{M^{\frac{3}{2}}}\,\Omega,\quad \frac{r}{\sigma} = \frac{\omega^3}{M^{\frac{3}{2}}}\,\nabla,$$

as may be shown without difficulty: and the angular velocity of the instantaneous axis is given by the equation $\varpi = \frac{M^{\frac{1}{2}}}{\omega^2}$; hence the relation between the two angular velocities is

$$\omega : \varpi = \frac{1}{\rho} - \frac{1}{\sigma} : \frac{1}{r}.$$