# 42.

## ON THE CAUSTIC BY REFLECTION AT A CIRCLE.

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THE following solution of the problem is that given by M. de St-Laurent (*Annales de Gergonne*, t. XVII. [1826] pp. 128—134); the process of elimination is somewhat different.

The centre of the circle being taken for the origin, let $k$ be its radius; $a$, $b$ the coordinates of the luminous point; $\xi$, $\eta$ those of the point at which the reflection takes place; $x$, $y$ those of any point in the reflected ray: we have in the first place

$$\xi^2 + \eta^2 = k^2 \quad \text{.......................................(1).}$$

There is no difficulty in finding the equation of the reflected ray[1]; this is

$$(b\xi - a\eta)(\xi x + \eta y - k^2) + (y\xi - x\eta)(a\xi + b\eta - k^2) = 0 \quad \text{................(2),}$$

[1] To do this in the simplest way, write

$$\rho^2 = (\xi - x)^2 + (\eta - y)^2, \quad \sigma^2 = (\xi - a)^2 + (\eta - b)^2,$$

then, by the condition of reflection,

$$\rho + \sigma = \text{min.},$$

$\rho$, $\sigma$ being considered as functions of the variables $\xi$, $\eta$, which are connected by the equation (1). Hence

$$\frac{\xi - x}{\rho} + \frac{\xi - a}{\sigma} + \lambda\xi = 0,$$

$$\frac{\eta - y}{\rho} + \frac{\eta - b}{\sigma} + \lambda\eta = 0;$$

or, eliminating $\lambda$,

$$\frac{\eta x - \xi y}{\rho} + \frac{\eta a - \xi b}{\sigma} = 0,$$

whence $\quad (\eta x - \xi y)^2 [(\xi - a)^2 + (\eta - b)^2] = (\eta a - \xi b)^2 [(\xi - x)^2 + (\eta - y)^2]$.

This may be written

$$\{(\eta x - \xi y)(\xi - a) - (\eta a - \xi b)(\xi - x)\}[(\eta x - \xi y)(\xi - a) + (\eta a - \xi b)(\xi - x)]$$
$$+ \{(\eta x - \xi y)(\eta - b) - (\eta a - \xi b)(\eta - y)\}[(\eta x - \xi y)(\eta - b) + (\eta a - \xi b)(\eta - y)] = 0;$$

the factors in { } reduce themselves respectively to $\xi P$ and $\eta P$, where $P = \xi(b - y) - \eta(a - x) + ay - bx$; omitting the factor $P$, (which equated to zero, is the equation of the line through $(a, b)$ and $(\xi, \eta)$,) and replacing $\xi(\xi - a) + \eta(\eta - b)$ and $\xi(\xi - x) + \eta(\eta - y)$ by $k^2 - a\xi - b\eta$ and $k^2 - \xi x - \eta y$, respectively, we have the equation given above.

or, arranging the terms in a more convenient order,

$$(bx+ay)(\xi^2-\eta^2)+2(by-ax)\xi\eta-k^2(b+y)\xi+k^2(a+x)\eta=0 \quad\text{.........(2')}.$$

Hence, considering $\xi$, $\eta$ as indeterminate parameters connected by the equation (1), the locus of the curve generated by the continued intersections of the lines (2) will be found by eliminating $\xi$, $\eta$, $\lambda$ from these equations and the system

$$\xi[\lambda+2(bx+ay)]+\eta[2(by-ax)]-k^2(b+y)=0 \quad\text{.................(3)},$$

$$\xi[2(by-ax)]+\eta[\lambda-2(bx+ay)]+k^2(a+x)=0 \quad\text{.................(4)},$$

and from these, multiplying by $\xi$, $\eta$, adding and reducing by (2), we have

$$-\xi(b+y)+\eta(a+x)-\lambda=0 \quad\text{.............................(5)},$$

which replaces the equation (2) or (2′). Thus the equations from which $\xi$, $\eta$, $\lambda$ are to be eliminated are (1), (3), (4), (5).

From (3), (4), (5), by the elimination of $\xi$, $\eta$, we have

$$\begin{aligned}-\lambda\{\lambda^2-4(bx+ay)^2\}&-4k^2(by-ax)(a+x)(b+y)\\ &-k^2(a+x)^2[\lambda+2(bx+ay)]\\ &-k^2(b+y)^2[\lambda-2(bx+ay)]\\ &+4\lambda(by-ax)^2=0 \quad\text{..........................................(6)},\end{aligned}$$

or, reducing,

$$\begin{aligned}-\lambda^3+\lambda\{4(a^2+b^2)(x^2+y^2)&-k^2[(a+x)^2+(b+y)^2]\}\\ &-2k^2(bx-ay)(x^2+y^2-a^2-b^2)=0 \quad\text{.............................(7)};\end{aligned}$$

which may be represented by

$$-\lambda^3+\lambda Q-2R=0 \quad\text{...................................(7')}.$$

Again, from the equations (4), (3), transposing the last terms and adding the squares, also reducing by (1),

$$\begin{aligned}k^4[(a+x)^2+(b+y)^2]&=k^2\lambda^2+4k^2(a^2+b^2)(x^2+y^2)\\ &+4\lambda[(\xi^2-\eta^2)(bx+ay)+2\xi\eta(by-ax)] \quad\text{.................(8)};\end{aligned}$$

but from the same equations, multiplying by $\xi$, $\eta$ and adding, also reducing by (1),

$$k^2\lambda+2(bx+ay)(\xi^2-\eta^2)+4\xi\eta(by-ax)+k^2[-\xi(b+y)+\eta(a+x)]=0 \quad\text{...........(9)},$$

or reducing by (5) and dividing by two,

$$k^2\lambda+(bx+ay)(\xi^2-\eta^2)+2\xi\eta(by-ax)=0 \quad\text{.......................(10)}.$$

Using this to reduce (8),

$$k^2 [(a+x)^2 + (b+y)^2] = 4 (a^2+b^2)(x^2+y^2) + 3\lambda^2 \quad\ldots\ldots\ldots\ldots\ldots(11),$$

or, from the value of $P$,

$$-3\lambda^2 + Q = 0 \quad\ldots\ldots\ldots\ldots\ldots\ldots(12),$$

which singularly enough is the derived equation of (7′) with respect to $\lambda$: so that the equation of the curve is obtained by expressing that two of the roots of the equation (7′) are equal. Multiplying (12) by $\lambda$ and reducing by (7′),

$$-\lambda Q + 3R = 0,$$

or, combining this with (12),

$$27R^2 - Q^3 = 0\,;$$

whence, replacing $R$, $Q$ by their values, we find

$$27k^4 (bx-ay)^2 (x^2+y^2-a^2-b^2)^2 - \{4(a^2+b^2)(x^2+y^2) - k^2 [(a+x)^2+(b+y)^2]\}^3 = 0,$$

the equation of M. de St-Laurent.