12.

ON THE INVOLUTION OF TWO MATRICES OF THE SECOND ORDER.

[*British Association Report, Southport* (1883), pp. 430—432.]

IF m, n be two matrices of any order i, then, taking the determinant of the matrix $z + yn + xm$, there results a ternary quantic in the variables x, y, z, which may be termed the quantic of the corpus m, n.

In what follows I confine myself almost exclusively to the case of a corpus of the second order; the quantic may be written

$$z^2 + 2bzx + 2cyz + dx^2 + 2exy + fy^2:$$

it is then easy to establish the identical relations

$$m^2 - 2bm + d = 0,$$
$$mn + nm - 2bn - 2cm + 2e = 0,$$
$$n^2 - 2cn + f = 0.$$

It hence easily appears that any given function of m, n can, by aid of the five parameters b, c, d, e, f, be expressed in the form $A + Bm + Cn + Dmn$.

This form containing four arbitrary constants, it follows that in general any given matrix of the second order can be expressed as a function of m and n; for there will be four linear equations between A, B, C, D and the four elements of the given matrix. But this statement is subject to two cases of exception.

The first of these is when n and m are functions of one another: for in this case $A + Bm + Cn + Dmn$ is reducible to the form $P + Qm$, and there will be only two disposable constants wherewith to satisfy the four linear equations.

The second case is when the determinant of the fourth order formed by the elements of the four matrices $\begin{vmatrix} 1, & m \\ n, & mn \end{vmatrix}$ vanishes; writing

$$m, n = \begin{vmatrix} t_1, & t_2 \\ t_3, & t_4 \end{vmatrix}, \quad \begin{vmatrix} \tau_1, & \tau_2 \\ \tau_3, & \tau_4 \end{vmatrix}$$

respectively, it is not difficult to show that the value of this determinant is

$$(t_2\tau_3 - \tau_2 t_3)^2 + \{(t_1 - t_4)\tau_2 - (\tau_1 - \tau_4)t_2\}\{(t_1 - t_4)\tau_3 - (\tau_1 - \tau_4)t_3\}.$$

This expression is a function of the five parameters b, c, d, e, f, as may be shown in a variety of ways.

Thus it is susceptible of easy proof that if μ_1, μ_2 are the roots of the equation $\mu^2 - 2b\mu + d = 0$, and ν_1, ν_2 the roots of the equation $\nu^2 - 2c\nu + f = 0$, then, the two matrices being related as above, we must have

$$(m - \mu_1)(n - \nu_1) = 0,$$
$$(n - \nu_2)(m - \mu_2) = 0,$$

and consequently, by virtue of the middle one of the three identities,

$$\mu_1\nu_1 + \mu_2\nu_2 - 2e = 0.$$

Writing this in the form

$$(\mu_1\nu_1 + \mu_2\nu_2 - 2e)(\mu_1\nu_2 + \mu_2\nu_1 - 2e) = 0,$$

this is $$4e^2 - 2e \,.\, 4bc + (\mu_1^2 + \mu_2^2)\nu_1\nu_2 + (\nu_1^2 + \nu_2^2)\mu_1\mu_2 = 0,$$

which gives $$e^2 - 2bce + b^2 f + c^2 d - df = 0\,;$$

the function on the left hand is the invariant (discriminant) of the ternary quantic appurtenant to the corpus, and we have this invariant $= 0$ as the necessary and sufficient condition of the involution of the elements of the corpus; the invariant in question is for this reason called the involutant.

Expressing the values of the coefficients in terms of the elements of the two matrices, namely

$$2b = t_1 + t_4, \quad 2c = \tau_1 + \tau_4,$$

$$d = t_1 t_4 - t_2 t_3, \quad 2e = t_1\tau_4 + \tau_1 t_4 - t_2\tau_3 - t_3\tau_2, \quad f = \tau_1\tau_4 - \tau_2\tau_3,$$

it at once appears that the two expressions for the involutant are, to a numerical factor *près*, identical.

It can be shown *à priori* that the involutant of a corpus of the second order must be expressible in terms of the coefficients of the function; and therefore, being obviously invariantive in regard to linear substitutions impressed on m, n, it must be also invariantive for linear substitutions impressed on z, x, y, and must therefore be the invariant of the function. The corresponding theorem is not true, it should be observed, for the involutant of a corpus beyond the second order; for such involutant cannot in general be expressed in terms of the coefficients of the function.

The expression for the involutant in terms of the t's and τ's may also be obtained directly from the equation $(m - \mu_1)(n - \nu_1) = 0$. To this end it is only necessary to single out any term of the matrix represented by the left-hand side of the equation and equate it to zero: the resulting equation rationalised will be found to reproduce the expression in question.

I have thus indicated four methods of obtaining the involutant to a matrix-corpus of the second order; but there is yet a fifth, the simplest of all, and the most suggestive of the course to be pursued in investigating the higher order of involutants.

I observe that for a corpus of any order the function $mn - nm$ is invariantive for any linear substitution impressed on m and n. Its determinant will therefore be an invariant for any substitution impressed on m and n. When m and n are of the second order, reducing each term of $(mn - nm)^2$, that is $mnmn - mn^2m - nm^2n + nmnm$, and of $mn - nm$, by means of the three identical equations, to the form of a linear function of mn, m, n, 1, it will be found without difficulty that there results the identical equation

$$(mn - nm)^2 + I = 0,$$

the coefficient of $mn - nm$ vanishing. Consequently the determinant of the matrix $mn - nm$ is equal to I, which on calculation will be found to be identical with the invariant of the ternary quadric function.

It is obvious from the three identical equations that if m, n are in involution—that is, if their involutant is zero—every rational and integral function of m, n will be in involution with every other rational and integral function of m, n. Hence follows this new and striking theorem concerning matrices of the second order: If $f(m, n)$ and $\phi(m, n)$ are any rational functions whatever of m, n, the determinant to the matrix $mn - nm$ is contained as a factor in the determinant to the matrix $f\phi - \phi f$.

It may be noticed that f, ϕ need not be integer functions *by stipulation*, because any linear function of mn, m, n, 1, divided anteriorly or posteriorly by a second like function, can itself be expressed as a linear function of the same four terms.

As a very simple example of the theorem, observe that the determinant of $m^2n - mnm$ will contain as a factor the determinant of $mn - nm$.