14.

ON QUATERNIONS, NONIONS, SEDENIONS, ETC.

[*Johns Hopkins University Circulars*, III. (1884), pp. 7—9.]

(1) SUPPOSE that m and n are two matrices of the second order.

Then if we call the determinant of the matrix $x + my + nz$,

$$x^2 + 2bxy + 2cxz + dy^2 + 2eyz + fz^2,$$

the necessary and sufficient conditions for the subsistence of the equation $nm = - mn$ is that $b = 0$, $c = 0$, $e = 0$, and if we superadd the equations $m^2 + 1 = 0$, $n^2 + 1 = 0$, then $d = 1$ and $f = 1$, or in other words in order to satisfy the equations $mn = - nm$, $m^2 = - 1$, $n^2 = - 1$, where it will of course be understood that in these (as in the equations $m^2 + 1 = 0$, $n^2 + 1 = 0$) 1 is the abbreviated form of the matrix $\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}$ and $\bar{1}$ of* the form $\begin{matrix} \bar{1} & 0 \\ 0 & \bar{1} \end{matrix}$, the necessary and sufficient condition is that the determinant of $x + my + nz$ shall be equal to $x^2 + y^2 + z^2$.

The simplest mode of satisfying this condition is to write $m = \begin{matrix} i & 0 \\ 0 & -i \end{matrix}$, $n = \begin{matrix} 0 & -1 \\ 1 & 0 \end{matrix}$, i meaning $\sqrt{(-1)}$, which gives $mn = \begin{matrix} 0 & -i \\ -i & 0 \end{matrix}$ and $nm = \begin{matrix} 0 & i \\ i & 0 \end{matrix}$.

It is easy to express any matrix of the second order as a linear function of 1 $\left(\text{meaning } \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}\right)$ m, n, p, where p stands for mn.

For if $\begin{matrix} a, & b \\ c, & d \end{matrix}$ be any such matrix it is only necessary to write

$$a = f + ig, \quad b = - h - ki,$$
$$d = f - ig, \quad c = - h + ki,$$

and then $\begin{matrix} a, & b \\ c, & d \end{matrix} = f + gm + hn + kp$.

The most general solution of the equations $MN = - NM$, $M^2 = N^2 = - 1$, must contain three arbitrary constants, namely, the difference between the number of terms in m and n, and the number of conditions $b = 0$, $c = 0$, $e = 0$, $d = 1$, $f = 1$, which are to be satisfied.

[* $\bar{1}$ denotes -1.]

Suppose M, N to be the most general solution fulfilling these conditions; we may write

$$M = f + gm + hn + kp,$$

$$N = f' + g'm + h'n + k'p,$$

where m, n is any particular solution and $p = mn$, and we shall have inasmuch as $M^2 = \bar{1}$,

$$f^2 - g^2 - h^2 - k^2 + 2fgm + 2fhn + 2fkp = \text{the matrix } \bar{1},$$

and consequently $$g^2 + h^2 + k^2 = 1 + f^2,$$

$$fg = 0, \quad fh = 0, \quad fk = 0.$$

Hence $f = 0$ and $$g^2 + h^2 + k^2 = 1.$$

Similarly $f' = 0$ and $$g'^2 + h'^2 + k'^2 = 1,$$

and also inasmuch as $MN = -NM$,

$$gg' + hh' + kk' = 0,$$

and since the equations $M^2 = \bar{1}$, $N^2 = \bar{1}$, $MN = -NM$ imply if we make $MN = P$ that $P^2 = -1$, and $MP = -PM$, and $NP = -PN$, it follows that M, N, P, are connected with m, n, p, in the same way as the coordinates of a point referred to one set of rectangular coordinates in space are connected with the coordinates of the same point referred to any other set of the same*.

Herein lies the ground of the geometrical interpretation to which quaternions lend themselves and it is hardly necessary to do more than advert to the fact that the theory of Quaternions is one and the same thing as that of Matrices of the second order viewed under a particular aspect†.

(2) Let m, n now denote matrices of the third order.

We might propose to solve the equation $mn = -nm$.

The result of the investigation is that we must have $m^2 = n^2$, $m^3 = 0$, $n^3 = 0$, and writing $mn = p$, $m^2 = n^2 = q$, there results a set of *quinions*, $1, m, n, p, q$, for which the multiplication is that marked (a_5) p. 144 of the late Prof. Peirce's invaluable memoir in Vol. IV. of the *American Journal of Mathematics*.

* There is another solution possible, obtained by writing

$$-\frac{f}{f'} = \frac{g}{g'} = \frac{h}{h'} = \frac{k}{k'}, \quad f^2 + g^2 + h^2 + k^2 = 0$$

but this leads to a linear relation between m and n, so that $mn = nm$ and consequently $mn = nm = 0$ which is not the kind of solution proposed in the question.

† See my article in the *Lond. and Edin. Phil. Mag.* on "Involution and Evolution of Quaternions," November, 1883. [Above, p. 112.]

But instead of this let us propose the equation $mn = \rho nm$, where ρ is one of the imaginary roots of unity; if now we write the determinant of $x + my + nz$ under the form

$$x^3 + 3bx^2y + 3cx^2z + 3dxz^2 + 6exyz + 3fyz^2 + gy^3 + 3hy^2z + 3kyz^2 + lz^3,$$

it may be shown [cf. p. 126, below] that we must have

$$b = 0, \quad c = 0, \quad d = 0, \quad e = 0, \quad f = 0, \quad h = 0, \quad k = 0,$$

and if we superadd the conditions $m^3 = 1$, $n^3 = 1$, we must also have $g = 1$, $l = 1$, or in other words the determinant to $x + my + nz$ must take the form $x^3 + y^3 + z^3$; but this condition (or system of conditions) although necessary *is not sufficient* (a point which I omitted to notice in my article entitled "A Word on Nonions" inserted* in a previous *Circular*).

It is obviously necessary that we must have $(mn)^3 = 1$.

Now if the *identical equation* to mn be written under the form

$$(mn)^3 - 3B(mn)^2 + 3Dmn - E = 0,$$

B may be shown to be a linear homogeneous function of b, c, and e; also $E = gl = 1$; but D is not a function of $b, c, d, e, f, g, h, k, l$, and will not in general vanish (as it is here required to do) when b, c, d, e, f, h, k vanish. Its value is the sum of the products obtained on multiplying each quadratic minor of m by its *altruistic* opposite in n: (the *proper* opposite to a minor of m means the minor which is the reflected image of such minor viewed in the Principal Diagonal of m regarded as a mirror; and the *altruistic* opposite is the minor which occupies in n a position precisely similar to that of the proper opposite in m). There are, therefore, 10 equations in all to be satisfied between the coefficients of m and n when $m^3 = n^3 = 1$ and $nm = \rho mn$.

These *ten* conditions I have demonstrated are *sufficient* as well as necessary. There remains then $18 - 10$ or 8 arbitrary constants in the general solution. If m, n is a particular solution we may take for M, N (the matrices of the general solution),

$$M = \alpha + \beta m + \gamma m^2 + \alpha' n + \beta' mn + \gamma' m^2 n + \alpha'' n^2 + \beta'' mn^2 + \gamma'' m^2 n^2,$$

$$N = \alpha_1 + \beta_1 m + \gamma_1 m^2 + \alpha_1' n + \beta_1' mn + \gamma_1' m^2 n + \alpha_1'' n^2 + \beta_1'' mn^2 + \gamma_1'' m^2 n^2,$$

and 10 relations between the 18 coefficients *must* be sufficient to enable to be satisfied the equations $M^3 = N^3 = 1$, $NM = \rho MN$: but what these relations are and how they may most simply be expressed I am not at present in a condition to state†.

[* Vol. III. of this Reprint, p. 647.]

† The solution of this problem would seem to involve some unknown expansion of the idea of orthogonalism. Unless $MN = NM = 0$, a solution to be neglected, it may be proved that $\alpha = 0$, $\alpha_1 = 0$.

I showed in "A Word on Nonions" that the 9 first conditions are satisfied by taking

$$m = \begin{matrix} 1 & 0 & 0 \\ 0 & \rho & 0 \\ 0 & 0 & \rho^2, \end{matrix} \qquad n = \begin{matrix} 0 & 0 & 1 \\ \rho & 0 & 0 \\ 0 & \rho^2 & 0. \end{matrix}$$

The 10th condition is also satisfied; for the only quadratic minors (not having a zero determinant) in m are $\begin{matrix} 1 & 0 \\ 0 & \rho \end{matrix}$, $\begin{matrix} \rho & 0 \\ 0 & \rho^2 \end{matrix}$, $\begin{matrix} 1 & 0 \\ 0 & \rho^2 \end{matrix}$; the *altruistic opposites* to which in n are $\begin{matrix} 0 & \rho \\ 0 & 0 \end{matrix}$, $\begin{matrix} 0 & \rho^2 \\ 0 & 0 \end{matrix}$, $\begin{matrix} 0 & 0 \\ 1 & 0 \end{matrix}$, the determinants to each of which are zeros, and accordingly we find

$$m^3 = n^3 = \begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1, \end{matrix}$$

$$nm = \begin{matrix} 0 & 0 & \rho^2 \\ \rho & 0 & 0 \\ 0 & 1 & 0, \end{matrix} \qquad mn = \begin{matrix} 0 & 0 & 1 \\ \rho^2 & 0 & 0 \\ 0 & \rho & 0, \end{matrix}$$

so that $mn = \rho nm$ and $m^3 = n^3 = 1$ as required.

I subjoin an outline proof of the fundamental portion of the theory of Quaternions and Nonions above stated as it will serve to throw much light upon the nature of the processes employed in that new world of thought to which I gave the name of Universal Algebra or the Algebra of multiple quantity: a fuller explanation will be found in the long memoir which I am preparing on the entire subject for the *American Journal of Mathematics*.

(1) As regards the equation $nm = -mn$, where m, n are matrices of the second order.

As before let the determinant of $(x + ym + zn)$ be

$$x^2 + 2bxy + 2cxz + dy^2 + 2eyz + fz^2.$$

I may observe here parenthetically that the Invariant of the above Quantic is equal to the determinant of $mn - nm$, and that when it vanishes $1, m, n, mn$, as also $1, n, m, nm$ are linearly related—or, as I express it, this Invariant is the Involutant of the system m, n or n, m. When m, n are of higher than the second order, the Involutant of m, n, say I, is that function whose vanishing implies that the 9 matrices $(1, m, m^2 \between 1, n, n^2)$ are linearly related, and the Involutant of n, m, say J, that function whose vanishing implies that the 9 quantities $(1, n, n^2 \between 1, m, m^2)$ are so related (I, J being two distinct functions), and so for matrices of any order higher than the second.

By virtue of a general theorem for any two matrices m, n of the second order, the following identities are satisfied:

$$m^2 - 2bm + d = 0,$$
$$mn + nm - 2bn - 2cm + 2e = 0,$$
$$n^2 - 2cn + f = 0.$$

If then $mn + nm = 0$, since m and n cannot be functions of one another (for then $mn = nm$), the second equation shows that $b = 0$, $c = 0$, $e = 0$, and conversely if $b = 0$, $c = 0$, $e = 0$, $mn + nm = 0$, and $m^2 + d = 0$, $n^2 + f = 0$, where, if we please, we may make $d = 1$, $f = 1$.

(2) Let m, n be matrices of the third order, and write as before,

$$\text{Det. } (x + ym + zn) = x^3 + 3bx^2y + 3cx^2z + 3dxy^2 + 6exyz + 3fxz^2 + gy^3 + 3hy^2z + 3kyz^2 + lz^3.$$

Then by virtue of the general theorem last referred to* there exist the identical equations

$$m^3 - 3bm^2 + 3dm - g = 0,$$
$$m^2n + mnm + nm^2 - 3b(mn + nm) - 3cm^2 + 3dn + 6em - 3h = 0,$$
$$mn^2 + nmn + n^2m - 3c(mn + nm) - 3bn^2 + 3fm + 6en - 3k = 0,$$
$$n^3 - 3cn^2 + 3fn - l = 0.$$

Let now $nm = \rho mn$, where ρ is either imaginary cube root of unity, then

$$(1)\ m^2n + mnm + nm^2 = 0 \text{ and } (2)\ mn^2 + nmn + n^2m = 0;$$

for greater simplicity suppose also that $m^3 = n^3 = 1$, where 1 means the matrix

$$\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}.$$

From the 1st and 2nd of the four identical equations combined it may be proved that $b = 0$, $d = 0$; I do not produce the proof here because to make it *rigorous*, the theory of Nullity would have to be gone into which would occupy too much space; and in like manner from the 3rd and 4th it may be shown that $c = 0$, $f = 0$†. Hence returning to the two middle equations it follows that $e = 0$, $h = 0$, $k = 0$, and from the two extremes that $g = 1$, $l = 1$.

If then $nm = \rho mn$, $m^3 = 1$, and $n^3 = 1$, it is necessary that

$$b = 0,\quad c = 0,\quad d = 0,\quad e = 0,\quad f = 0,\quad g = 1,\quad h = 0,\quad k = 0,\quad l = 1.$$

But these equations although necessary are manifestly insufficient; for they lead to the equations $m^3 - 1 = 0$, $n^3 - 1 = 0$, and

$$(1)\ m^2n + mnm + nm^2 = 0;\ (2)\ mn^2 + nmn + n^2m = 0,$$

[* By Cayley's theorem, if in Det. $(x + ym + zn)$ we replace x by $-ym - zn$, the result vanishes identically in regard to y and z.]

† Except when m, n are functions of one another, so that mn and nm are identical and consequently are each of them zero.

but not necessarily to $nm = \rho mn$. In fact the supposed equations between m and n involve as a consequence the equation $(mn)^3 = 1$. Now the general identical equation to (mn) is

$$(mn)^3 - 3B(mn)^2 + 3D(mn) - F = 0,$$

where B is the sum of each term in m by its altruistic opposite in n $= 3bc - 2e = 0$, $F = gl = 1$, and D is the sum of each first minor in m by its altruistic opposite in n which sum does not necessarily vanish when b, c, d, e, f, h, k, all vanish. Hence there is a 10th condition necessary not involved in the other 9, namely, $D = 0$. These 10 conditions I shall show are sufficient as well as necessary. For when they are satisfied since $(mn)^3 = 1$, $mn \,.\, mn = n^2m^2$.

Hence from (1) $$m^2n^2 + n^2m^2 + nm^2n = 0,$$
and from (2) $$m^2n^2 + n^2m^2 + mn^2m = 0.$$

Hence $nm \,.\, mn = mn \,.\, nm$ *, and consequently nm is a function of mn [cf. p. 149, below]. Hence we may write

$$nm = A + Bmn + C(mn)^2.$$

But the *latent roots* of mn and nm (which are always identical) are $1, \rho, \rho^2$, hence

$$A + B + C, \quad A + B\rho + C\rho^2, \quad A + B\rho^2 + C\rho,$$

must be equal to $1, \rho, \rho^2$, each to each taken in some one of the 6 orders in which these quantities can be written†.

Solving these 6 systems of linear equations there results:

$$A = 0, \qquad B = 0, \qquad C = 1, \ \rho \ \text{or} \ \rho^2$$

or $$A = 0, \qquad B = 1, \ \rho \ \text{or} \ \rho^2, \qquad C = 0.$$

Hence $nm = \theta mn$, or $\theta(mn)^2$ where $\theta = 1, \rho, \rho^2$.

If $$nm = \theta(mn)^2, \quad nmmn = \theta(mn)^3 = \theta.$$

Hence $$m^2 = \theta n^2 \,.\, \theta n^2 = \theta^2 n;$$

and $$m^2n + mnm + nm^2 = 3\theta m^4 = 3\theta m = 0,$$

so that $m = 0$, and $m^3 = 0 = 1$; and again if $nm = mn$,

$$m^2n + mnm + nm^2 = 2m^2n + mnm = 3m^2n = 0,$$

* This equation is independent of the equation $(mn)^3 = 1$; for

$$nm^2n - mn^2m = (m^2n + mnm + nm^2)n - m(mn^2 + nmn + n^2m) = 0$$

by virtue of equations (1) and (2) above: accordingly these equations taken alone imply the equations

$$nm = A + B_1 mn + C(mn)^2, \quad mn = -A + B_2 nm - C(nm)^2$$

where B_1, B_2 are the roots of $B^2 + B + 1 - \frac{AC}{2} = 0$; A, C being arbitrary and independent except that each vanishes when and only when the cube of mn and (as a consequence) of nm, is a scalar matrix. [See below, p. 154. Footnote [†].]

† By virtue of the general theorem that the latent roots of any function of a matrix are the like functions of the latent roots of the original matrix.

so that $m^3n = 0$, $n = 0$, and $n^3 = 0 = 1$ as before, where it should be noticed

that $0 = 1$ means that $\begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix}$ is identical with $\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}$.

Hence the only available hypothesis remaining is the equation $nm = v \,.\, mn$, where v is one of the imaginary cube-roots of unity as was to be proved.

(3) It remains to say a few words on the general equation $nm = kmn$, where m, n are matrices of any order ω. To avoid prolixity I shall confine my remarks to the general case, which is, that where the determinants (or as I am used to say the contents) of m and n are each of them finite; with this restriction, the proposed equation is impossible for general values of k as will be at once obvious from the fact that the totalities of the latent roots of mn and of nm are always identical, but the individual latent roots are by virtue of the proposed equation in the ratio to one another of $1 : k$, which, since by hypothesis no root is zero, is only possible when $k^\omega = 1$.

When the above equation is satisfied the ω^2 equations arising from the identification of nm with kmn cease to be incompatible and (as is necessary or at all events usual in such a contingency) become mutually involved. Thus, for example, when $\omega = 1$ and $k = 1$, the number of independent equations is 0, that is, $1 - 1$, when $\omega = 2$ and $k = -1$ the number is 3, that is, $4 - 1$, when $\omega = 3$ and $k = \rho$ or ρ^2 the number is 8, that is, $9 - 1$; it is fair therefore to presume (although the assertion requires proof) that for any value of ω when k is a primitive ωth root of unity the number of conditions to be satisfied when $nm = kmn$ is $\omega^2 - 1$. Of these the condition that the content of $x + my + nz$ shall be of the form $x^\omega + cy^\omega + c'z^\omega$ will supply

$$\frac{(\omega+1)(\omega+2)}{2} - 3, \text{ that is, } \frac{\omega^2 + 3\omega}{2} - 2,$$

and there will therefore be

$$\frac{\omega^2 - 3\omega}{2} + 1 \text{ or } \frac{(\omega-1)(\omega-2)}{2}$$

to be supplied from some other source.

When k is a non-primitive ωth root of unity, the number of equations of condition is no longer the same. Thus when $k = 1$ we know that n may be of the form

$$A + Bm + Cm^2 + \ldots + Lm^{\omega-1},$$

where A, B, ... L, and all the ω^2 terms in m are arbitrary, and consequently the number of conditions for that case is $2\omega^2 - (\omega^2 + \omega)$ or $\omega^2 - \omega$. It seems then very probable that if k is a qth power of a primitive ωth root of unity the number of conditions required to satisfy $nm = kmn$ is $\omega^2 - \delta$ where δ is the greatest common measure of q and ω: but, of course, this assertion awaits confirmation.

When $\omega = 4$ besides the case of $nm = mn$, that is, of n being a function of m of which the solution is known, there will be two other cases to be considered, namely, $nm = -mn$ and $nm = imn$: the former probably requiring 14 and the latter 15 conditions to be satisfied between the coefficients of m, the coefficients of n and the two sets of coefficients combined.

It is worthy of notice that the conditions resulting from the content of $x + my + nz$ becoming a sum of 3 powers are incompatible with the equation $nm = vmn$ when v is other than a primitive ωth root of unity (ω being of course the order of m or n).

Thus suppose $\omega = 4$; the conditions in question applied to the middle one of the 5 identical equations give

$$m^2n^2 + n^2m^2 + mn^2m + nm^2n + mnmn + nmnm = 0;$$

when $nm = imn$ the left-hand side of this equation becomes

$$(1 + i^4 + i^2 + i^2 + i + i^3)\, m^2n^2,$$

that is, is zero, but when $nm = -mn$, the value is

$$(1 + 1 - 1 - 1 - 1 - 1)\, m^2n^2$$

which is not zero, and so in general. Thus the pure power form of the content of $x + my + nz$ is a condition applicable to the case of $\frac{nm}{mn}$ being a primitive root of unity and to no other.

The case of nm being a primitive root of ordinary unity is therefore the one which it is most interesting to thrash out.

There are in this case, we have seen, $\frac{1}{2}(\omega^2 + 3\omega - 4)$ simple conditions expressible by the vanishing of that number of coefficients in the content of $x + my + nz$ and $\frac{1}{2}(\omega - 1)(\omega - 2)$ supplemental ones. What are these last? I think their constitution may be guessed at with a high degree of probability. For revert to the case of $\omega = 3$ in which there is one such found by equating to zero the second coefficient in the identical equation

$$(mn)^3 - 3B(mn)^2 + 3Dmn - G = 0.$$

Suppose now $\quad (m^2n^2)^3 - 3B'(m^2n^2)^2 + 3D'm^2n^2 - G' = 0$

is the identical equation to m^2n^2. By virtue of the 8 conditions supposed to be satisfied we *know* that $nm = \rho mn$ as well as $m^3 = 1$, $n^3 = 1$, and consequently that $(m^2n^2)^3 = 1$. Hence $B' = 0$, $D' = 0$, by virtue of the 7 parameters in the oft-quoted content and of D being all zero, and thus the evanescence of B' or D' imports no new condition.

Now suppose $\omega = 4$, and that

$$(mn)^4 - 4B(mn)^3 + 6D(mn)^2 - 4Gmn + M = 0,$$
$$(m^2n^2)^4 - 4B'(m^2n^2)^3 + 6D'(m^2n^2)^2 - 4G'm^2n^2 + M' = 0.$$

Here we know that B vanishes by virtue of b, c and e vanishing, but $D=0$, $G=0$, which must be satisfied if $nm = imn$, will be two new conditions not implied in those which precede. It seems then, although not certain, highly probable that $B'=0$, $D'=0$, will be implied in the satisfaction of the antecedent conditions but that $G'=0$ will be an independent condition, so that $D=0$, $G=0$, $G'=0$, will be the three supplemental conditions: and again when $\omega=5$ forming the identical equations to mn, m^2n^2, m^3n^3, and using an analogous litteration to what precedes, the supplemental conditions will be

$$\begin{array}{lll} D=0, & G=0, & M=0, \\ & G'=0, & M'=0, \\ & & M''=0, \end{array}$$

and so in general for any value of ω.

The functions D, G, M, etc., above equated to zero are known from the following theorem of which the proof will be given in the forthcoming memoir*.

If $$(\overline{mn})^\omega + k_1(\overline{mn})^{\omega-1} + \ldots + k_i(\overline{mn})^{\omega-i} + \ldots = 0$$

is the *identical equation* to mn, then k_i is equal to the sum of the product of each minor of order i in m multiplied by its *altruistic* opposite in n.

The annexed example will serve to illustrate in the case of $\omega = 3$ that unless the supplemental condition is satisfied we cannot have $nm = \rho mn$.

Write
$$m = \begin{matrix} 1 & 0 & 0, \\ 0 & \rho & 0, \\ 0 & 0 & \rho^2, \end{matrix} \qquad n = \begin{matrix} 0 & c & k, \\ k & 0 & c\rho, \\ c\rho^2 & k & 0, \end{matrix}$$

then the determinant to $x + my + nz$ will be easily found to be

$$x^3 + y^3 + (c^3 + k^3)z^3;$$

but D becomes $-3\rho ck$, and does not vanish unless $c=0$ or $k=0$, and accordingly we find

$$nm = \begin{matrix} 0 & \rho c & \rho^2 k, \\ k & 0 & c, \\ \rho^2 c & \rho k & 0, \end{matrix} \qquad mn = \begin{matrix} 0 & c & k, \\ \rho k & 0 & \rho^2 c, \\ \rho c & \rho^2 k & 0. \end{matrix}$$

When $k=0$ $mn = \rho^2 nm$, when $c=0$ $nm = \rho^2 mn$, but on no other supposition will $\frac{nm}{mn}$ be a primitive cube root of unity.

* This theorem furnishes as a Corollary the principle employed to prove the stability of the Solar System. (See *Lond. and Edin. Phil. Mag.*, October, 1883.) [Above, p. 110.]

ADDENDUM.

Referring to the equation $MN = -NM$, and to the eight equations expressing M and N in terms of the combinations of the powers of m with those of n, in which it is to be understood that M and N are *non-vacuous*, we know that the sums of the latent roots of M and of N must each vanish and consequently, as may be proved, that $a=0$, $a'=0$, leaving $8-2$ or 6 conditions to be satisfied. If we further stipulate that $M^3=1$, $N^3=1$, there will be 8 relations connecting the coefficients $b, c, \ldots k$ and $b', c', \ldots k'$, so that the 64 coefficients in the 8 equations connecting M, M^2; N, N^2; MN, M^2N^2; M^2N, MN^2, or say rather M, M^2; N, N^2; $\rho^2 MN, \rho^2 M^2N^2$; $\rho M^2 N, \rho MN^2$, with like combinations or multiples of combinations of powers of m, n* will be connected together by 56 equations; the coefficients in the expression for any one of the above 8 terms may then be arranged in pairs f_i, f_i'; g_i, g_i'; h_i, h_i'; k_i, k_i'; and in the expression for its fellow by F_i, F_i'; G_i, G_i'; H_i, H_i'; K_i, K_i'; so that the Matrix is resolved as it were into 4 sets of paired columns and 4 sets of paired lines; the 4 different sets of paired lines being found by writing successively $i=1, 2, 3, 4$.

It is then easy to see that there will be 4 equations of the form

$$\Sigma(f_\alpha G_\alpha' + f_\alpha G_\alpha') = 1,$$

and 6 quaternary groups (that is, 24 equations) of the form

$$\Sigma(f_\alpha G_\beta' + f_\alpha G_\beta') = 0,$$

with liberty to change f into F or G into g or each into each: together then the above are 28 of the 56 conditions required. But inasmuch as the 8 $[m, n]$ arguments may be interchanged with the 8 $[M, N]$ ones, we may transform the above equations by substituting for each letter f its conjugate $\frac{d \log \Delta}{df}$ (where Δ is the content of the Matrix) and thus obtain 28 others, giving in all (if the two sets as presumably is the case are independent) the required 56 conditions: the latter 28, however, may be replaced by others of much simpler form†.

* It is easy to see that the sum of the latent roots of M^iN^j must be zero for all values of i, j so that it is a homogeneous linear function of the 8 quantities $m, m^2, \ldots, mn, m^2n^2$.

† I am still engaged in studying this matrix, which possesses remarkable properties. Is it orthogonal? I rather think not, but that it is allied to a system of 4 pairs of somethings drawn in four mutually perpendicular hyperplanes in space of 4 dimensions. In the general case of $MN=\rho NM$ where ρ is a primitive ωth root of unity, there will be an analogous matrix of the order ω^2-1 where each line and each column will consist of $\omega+1$ groups of $\omega-1$ associated terms.

The value of the cube of any one of the 8 matrices M, M^2; ...; MN, M^2N^2 may be expressed as follows: It is P into ternary unity. Such a quantity may be termed by analogy a Scalar. To find $P_{i,j}$ I imagine the 8 letters corresponding to M^iN^j (but without powers of ρ attached) to be set over 8 of the 9 points of inflexion to any cubic curve, the paired letters being made suitably

To me it seems that this vast new science of multiple quantity soars as high above ordinary or quaternion Algebra as the Mécanique Céleste above the "Dynamics of a Particle" or a pair of particles, (if a new Tait and Steele should arise to write on the Dynamics of such pair,) and is as well entitled to the name of Universal Algebra as the Algebra of the past to the name of Universal Arithmetic.

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collinear with the missing 9th point. Then among themselves the 8 letters may be taken in 8 different ways to form collinear triads and the product of the letters in each triad may be called a collinear product; $P_{i,j}$ (which is identical with the Determinant to M^iN^j) will be the sum of the cubes of the 8 letters less 3 times the sum of their 8 collinear products, and its 8 values will be analogous to the 3 values of the sum of 3 squares in the Quaternion Theory. Each of these 8 values is assumed equal to unity.

It may be not amiss to add that the product of four squares by four is representable rationally as a sum of four squares, so if we place (not now 8 specially related but) nine perfectly arbitrary letters over the nine points of inflexion of a cubic curve the sum of their 9 cubes less three times their 12 collinear products multiplied by a similar function of 9 other letters may be expressed by a similar function of 9 quantities lineo-linear functions of the two preceding sets of 9 terms.

By the 8 letters of any set as, for example, $b, \ldots, h'$ being "specialized," I mean that they are subject to the condition $bb'+dd'+ff'+hh'=0$. When this equation is satisfied, and not otherwise, M^3 will be a Scalar, and it *must* be satisfied when $MN=\rho NM$.