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NOTE IN ILLUSTRATION OF CERTAIN GENERAL THEOREMS OBTAINED BY DR LIPSCHITZ.

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THE paper by Dr Lipschitz, which follows the present Note [in the *Quarterly Journal*, *l.c.*], is supplemental to Memoirs by him in *Crelle*, vols. LXX., LXXII., and LXXIV.; and he makes use of certain theorems obtained by him in these memoirs; these theorems may be illustrated by the consideration of a particular example.

Imagine a particle not acted on by any forces, moving in a given surface; and let its position on the surface at the time t be determined by means of the general coordinates x, y. We have then the vis-viva function T, a given function of x, y, x', y'; and the equations of motion are

$$\frac{d}{dt}\frac{dT}{dx'}-\frac{dT}{dx}=0,\quad \frac{d}{dt}\frac{dT}{dy'}-\frac{dT}{dy}=0,$$

which equations serve to determine x, y in terms of t, and of four arbitrary constants; these are taken to be the initial values (or values corresponding to the time $t=t_0$) of x, y, x', y'; say the values are α, β, α', β'.

We have the theorem that x, y are functions of α, β, $\alpha'(t-t_0)$, $\beta'(t-t_0)$.

Suppose for example that x, y, z denote ordinary rectangular coordinates, and that the particle moves on the sphere $x^2+y^2+z^2=c^2$; to fix the ideas, suppose that the coordinates z are measured vertically upwards, and that the particle is on the upper hemisphere; that is, take $z=+\sqrt{(c^2-x^2-y^2)}$, we have

$$T=\tfrac{1}{2}(x'^2+y'^2+z'^2),$$

where z' denotes its value in terms of x, y, x', y'; viz. we have $xx'+yy'+zz'=0$, or

$$z'=-\frac{xx'+yy'}{z},\quad =-\frac{xx'+yy'}{\sqrt{(c^2-x^2-y^2)}};$$

the proper value of T is thus

$$= \tfrac{1}{2}\left\{x'^2 + y'^2 + \frac{(xx' + yy')^2}{c^2 - x^2 - y^2}\right\},$$

but it is convenient to retain z, z', taking these to signify throughout their foregoing values in terms of x, y, x', y'.

The constants of integration are, as before, α, β, α', β'; but we use also γ, γ' considered as signifying given functions of these constants, viz. we have

$$\gamma = \sqrt{(c^2 - \alpha^2 - \beta^2)} \text{ and } \gamma' = -\frac{\alpha\alpha' + \beta\beta'}{\sqrt{(c^2 - \alpha^2 - \beta^2)}},$$

(in fact, $\alpha^2 + \beta^2 + \gamma^2 = c^2$ and $\alpha\alpha' + \beta\beta' + \gamma\gamma' = 0$; γ, γ' being thus the initial values of z, z').

Now, writing

$$\sigma = \frac{(t - t_0)\sqrt{(\alpha'^2 + \beta'^2 + \gamma'^2)}}{c},$$

the required values of x, y and the corresponding value of z are

$$x = \alpha \cos\sigma + \frac{c\alpha'}{\sqrt{(\alpha'^2 + \beta'^2 + \gamma'^2)}}\sin\sigma,$$

$$y = \beta \cos\sigma + \frac{c\beta'}{\sqrt{(\alpha'^2 + \beta'^2 + \gamma'^2)}}\sin\sigma,$$

$$z = \gamma \cos\sigma + \frac{c\gamma'}{\sqrt{(\alpha'^2 + \beta'^2 + \gamma'^2)}}\sin\sigma.$$

To verify that these are functions of α, β, $\alpha'(t - t_0)$, $\beta'(t - t_0)$, write $\alpha'(t - t_0) = u$, $\beta'(t - t_0) = v$; and take also $\gamma'(t - t_0) = w$; we have $\alpha u + \beta v + \gamma w = 0$, viz. $w, = -\frac{1}{\gamma}(\alpha u + \beta v)$, is a function of α, β, u, v; and then

$$\sigma = \frac{\sqrt{(u^2 + v^2 + w^2)}}{c},$$

and

$$x = \alpha\cos\sigma + \frac{u}{\sigma}\sin\sigma,$$

$$y = \beta\cos\sigma + \frac{v}{\sigma}\sin\sigma,$$

$$z = \gamma\cos\sigma + \frac{w}{\sigma}\sin\sigma;$$

so that x, y (and also z) are each of them a function of α, β, u, v, that is α, β, $\alpha'(t - t_0)$, $\beta'(t - t_0)$, which is the theorem in question.

The original variables are x, y; the quantities $\alpha'(t - t_0)$, $\beta'(t - t_0)$, or u, v are Dr Lipschitz' "Normal-Variables," and the theorem is that the original variables are functions of their initial values, and of the normal-variables.

The vis-viva function T may be expressed in terms of the normal-variables and their derived functions; viz. it is easy to verify that we have

$$T = \tfrac{1}{2}\left(\frac{1}{c^2\sigma^2} - \frac{\sin^2\sigma}{c^2\sigma^4}\right)(uu' + vv' + ww')^2$$

$$- \tfrac{1}{2}\frac{\sin^2\sigma}{\sigma}(u'^2 + v'^2 + w'^2),$$

where w denotes $-\frac{1}{\gamma}(\alpha u + \beta v)$ and consequently w' denotes $-\frac{1}{\gamma}(\alpha u' + \beta v')$; introducing herein differentials instead of derived functions, or writing

$$\phi(du) = \tfrac{1}{2}\left(\frac{1}{c^2\sigma^2} - \frac{\sin^2\sigma}{c^2\sigma^4}\right)(udu + vdv + wdw)^2$$

$$+ \tfrac{1}{2}\frac{\sin^2\sigma}{\sigma^2}(du^2 + dv^2 + dw^2),$$

where w, dw denote $-\frac{1}{\gamma}(\alpha u + \beta v)$, $-\frac{1}{\gamma}(\alpha du + \beta dv)$ respectively; then $\phi(du)$ is the function thus denoted by Dr Lipschitz: and writing herein $t - t_0 = 0$, and thence $u = 0$, $v = 0$, $w = 0$, $\sigma = 0$, the resulting value of $\phi(du)$ is

$$f_0(du), \quad = \tfrac{1}{2}(du^2 + dv^2 + dw^2),$$

where $f_0(du)$ is the function thus denoted by him; the corresponding value of $f_0(u)$ is $= \tfrac{1}{2}(u^2 + v^2 + w^2)$. We have thus an illustration of his theorem that the function $\phi(du)$ is such that we have identically

$$\phi(du) - \{d\sqrt{\{f_0(u)\}}\}^2 = \frac{m^2}{2f_0(u)}[f_0(du) - \{d\sqrt{(f_0 u)}\}^2],$$

where m is a function of u, v independent of the differentials du, dv; the value in the present example is in fact $m^2 = c^2\sin^2\sigma$; or the identity is

$$\phi(du) - \{d\sqrt{(f_0 u)}\}^2 = \frac{c^2\sin^2\sigma}{2(f_0 u)}[f_0(du) - \{d\sqrt{(f_0 u)}\}^2],$$

in verification whereof observe that we have

$$d\sqrt{(f_0 u)} = \frac{df_0(u)}{2\sqrt{(f_0 u)}} = \frac{udu + vdv + wdw}{\sqrt{(u^2 + v^2 + w^2)}}$$

$$= \frac{1}{c\sigma}(udu + vdv + wdw)^2.$$

The value of the left-hand side is thus

$$= -\frac{\sin^2\sigma}{c^2\sigma^4}(udu + vdv + wdw)^2 + \tfrac{1}{2}\frac{\sin^2\sigma}{\sigma^2}(du^2 + dv^2 + dw^2),$$

viz. this is

$$= \frac{c^2\sin^2\sigma}{c^2\sigma^2}\left\{\tfrac{1}{2}(du^2 + dv^2 + dw^2) - \frac{1}{c^2\sigma^2}(udu + vdv + wdw)^2\right\};$$

or, finally, it is

$$= \frac{c^2\sigma^2}{2f_0(u)}\left\{f_0(du) - [d\sqrt{\{f_0(u)\}}]^2\right\},$$

which is right.