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NOTE ON THE THEORY OF PRECESSION AND NUTATION.

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We have in the dynamical theory of Precession and Nutation (see Bessel's *Fundamenta* (1818), p. 126),

$$C\frac{dp}{dt}+(B-A)\,qr=LS\,(x'y-xy')\,dm'\left(\frac{1}{\Delta^3}-\frac{1}{r^3}\right),$$

$$A\frac{dq}{dt}+(C-B)\,rp=LS\,(y'z-yz')\,dm'\left(\frac{1}{\Delta^3}-\frac{1}{r^3}\right),$$

$$B\frac{dr}{dt}+(A-C)\,pq=LS\,(z'x-zx')\,dm'\left(\frac{1}{\Delta^3}-\frac{1}{r^3}\right),$$

where L is the mass of the Sun or Moon, x, y, z the coordinates of its centre referred to the centre of the Earth as origin,

$$r=\sqrt{x^2+y^2+z^2},$$

the distance of its centre, and

$$\Delta=\sqrt{(x-x')^2+(y-y')^2+(z-z')^2},$$

the distance of its centre from an element dm', coordinates (x', y', z') of the Earth's mass, the sum or integral S being extended to the whole mass of the Earth—I have written dm', r for Bessel's dm, r_1—, we have

$$\Delta^2=r^2-2\,(xx'+yy'+zz')+x'^2+y'^2+z'^2;$$

and thence

$$\frac{1}{\Delta^3}-\frac{1}{r^3}=\frac{3}{r^5}(xx'+yy'+zz')-\tfrac{3}{2}\frac{1}{r^7}\{(x^2+y^2+z^2)(x'^2+y'^2+z'^2)-5\,(xx'+yy'+zz')^2\}+\text{etc.}$$

The principal term is the first one,

$$\frac{3}{r^5}(xx' + yy' + zz');$$

but Bessel takes account also of the second term,

$$-\tfrac{3}{2}\frac{1}{r^7}\{(x^2 + y^2 + z^2)(x'^2 + y'^2 + z'^2) - 5(xx' + yy' + zz')^2\},$$

viz. considering the Earth as a solid of revolution (as to density as well as exterior form), he obtains in regard to it the following terms of $\sin\omega\frac{d\psi}{dt}$ and $\frac{d\omega}{dt}$ respectively;

$$\frac{3L}{4r^4}\,\frac{1}{Cn}\,.\,2(C-A)K(5\sin^2\delta - 1)\cos\delta\sin\alpha,$$

$$-\frac{3L}{4r^4}\,\frac{1}{Cn}\,.\,2(C-A)K(5\sin^2\delta - 1)\cos\delta\cos\alpha,$$

where

$$2(C-A)K = S(3\mu - 5\mu^3)\,2\pi\rho\,R^5dR\,d\mu,$$

K being in fact a numerical quantity, relating to the Earth only, and the value of which is by pendulum observations ultimately found to be $= 0{\cdot}13603$.

Writing, for shortness,

$$(x^2 + y^2 + z^2)(x'^2 + y'^2 + z'^2) - 5(xx' + yy' + zz')^2 = \Omega,$$

then the foregoing terms of $\sin\omega\frac{d\psi}{dt}$ and $\frac{d\omega}{dt}$ depend, as regards their form, on the theorem that for any solid of revolution (about the axis of z) we have

$$S(x'y - xy')\,\Omega dm',\quad S(y'z - yz')\,\Omega dm',\quad S(z'x - zx')\,\Omega dm'$$

$$= 0,$$

$$\tfrac{1}{2}y(x^2 + y^2 + z^2 - 5z^2)\,S[3(x'^2 + y'^2 + z'^2) - 5z'^2]\,z'dm',$$

$$-\tfrac{1}{2}x(x^2 + y^2 + z^2 - 5z^2)\,S[3(x'^2 + y'^2 + z'^2) - 5z'^2]\,z'dm',$$

respectively: viz. writing $x'^2 + y'^2 + z'^2 = R^2$, and $z' = R\mu$, also $x^2 + y^2 + z^2 = r^2$ and $x = r\cos\delta\cos\alpha$, $y = r\cos\delta\sin\alpha$, $z = r\sin\delta$, the values would be

$$0,$$

$$\tfrac{1}{2}r^3\cos\delta\sin\alpha(1 - 5\sin^2\delta)\,S(3 - 5\mu^2)\,\mu R^3dm',$$

$$-\tfrac{1}{2}r^3\cos\delta\sin\alpha(1 - 5\sin^2\delta)\,S(3 - 5\mu^2)\,\mu R^3dm',$$

which are of the form in question.

The verification is easy: the solid being one of revolution about the axis of z, any integral such as $Sx'z'^2dm'$ or $Sx'y'z'dm'$ which contains an odd power of x' or of y' is $= 0$; while such integrals as $Sx'^2z'dm'$, $Sy'^2z'dm'$ are equal to each other, or, what is the same thing, each $= \tfrac{1}{2}S(x'^2 + y'^2)\,z'dm'$. That we have $S(x'y - xy')\,\Omega dm' = 0$ is at once seen to be true; considering the next integral $S(y'z - yz')\,\Omega dm'$, the terms of $(y'z - zy')\,\Omega$ which lead to non-evanescent integrals are

$$-yz'\,.\,(x^2+y^2+z^2)(x'^2+y'^2+z'^2),$$
$$-5y'z\,.\,2yzy'z',$$
$$+5yz'\,.\,(x^2x'^2+y^2y'^2+z^2z'^2);$$

giving in the integral the several terms

$$-y\,(x^2+y^2+z^2)\,S\,(x'^2+y'^2+z'^2)\,z'dm',$$
$$-10yz^2\,.\,\tfrac{1}{2}\,S\,(x'^2+y'^2+z'^2-z'^2)\,z'dm',$$
$$+5y\,(x^2+y^2+z^2-z^2)\,.\,\tfrac{1}{2}\,S\,(x'^2+y'^2+z'^2-z'^2)\,z'dm',$$
$$+yz^2\,Sz'^3\,dm',$$

viz. collecting, the value is

$$(-1+\tfrac{5}{2}=)\quad \tfrac{3}{2}\,(x^2+y^2+z^2)\,yS\,(x'^2+y'^2+z'^2)\,z'dm',$$
$$(-\tfrac{5}{2}=)-\tfrac{5}{2}\,(x^2+y^2+z^2)\,y\Sigma z'^3dm',$$
$$(-\tfrac{5}{2}-5=)-\tfrac{15}{2}\,yz^2S\,(x'^2+y'^2+z'^2)\,z'dm',$$
$$(+\tfrac{5}{2}+5+5=)+\tfrac{25}{2}\,yz^2Sz'^3dm';$$

which is

$$=\quad \tfrac{1}{2}\,y\,(x^2+y^2+z^2-5z^2)\,S\,[3\,(x'^2+y'^2+z'^2)-5z'^2]\,z'dm';$$

and similarly the last term is

$$=-\tfrac{1}{2}\,x\,(x^2+y^2+z^2-5z^2)\,S\,[3\,(x'^2+y'^2+z'^2)-5z'^2]\,z'dm',$$

which completes the proof.