Arch. Mech., 37, 4–5, pp. 521—534, Warszawa 1985

# Anisotropic damage modelling for brittle elastic materials

F. SUPARTONO and F. SIDOROFF (LYON)

A GENERAL mechanical and thermodynamical framework for the description of anisotropic damage in elastic brittle materials is developed. A second order symmetric damage tensor is introduced and the damage elastic law resulting from an energetic identification is briefly recalled. Evolution laws for the damage tensor are then derived within the framework of generalized standard materials and starting from a damage criterion. Identification of the model from a tensile test is described and the resulting behaviour is exemplified in some plane stress situations. As a conclusion dissymmetrization of the behaviour between tension and compression is discussed.

Przedstawiono ogólne mechaniczne i termodynamiczne podstawy opisu zniszczenia anizotropowego w materiałach sprężysto-kruchych. Wprowadzono symetryczny tensor zniszczenia drugiego rzędu i omówiono krótko prawo zniszczenia wynikające z identyfikacji energetycznej. Prawa ewolucji tensora zniszczenia wyprowadzono dla uogólnionego materiału standardowego wychodząc z kryterium zniszczenia. Opisano identyfikację modelu na podstawie próby rozciągania ilustrując wyniki przykładami płaskiego stanu naprężenia. Przedyskutowano problem utraty symetrii między ściskaniem i rozciąganiem.

Представлены общие механические и термодинамические основы описания анизотропного разрушения в упруго-хрупких материалах. Введен симметричный тензор разрушения второго порядка и кратко обсужден закон разрущения, вытекающий из энергетической идентификации. Закон эволюции тензора разрушения выведен для обобщенного стандартного материала, исходя из критерия разрушения. Описана идентификация модели на основе испытания растяжения, иллюстрируя результаты примерами плоского напряженного состояния. Обсуждена проблема потери симметрии между сжатием и растяжением.

## 1. Introduction

THE BEHAVIOUR of concrete in tension and compression shows some typical features which are summarized (and idealized) in Fig. 1.

Essentially, the behaviour in tension can be described as brittle elastic while compression induces more complex coupling between elasticity, plasticity and brittle fracture. In any case, the elastic range in tension is much smaller than in compression. Starting from these features, there are two good reasons for modelling concrete within the framework of damage mechanics:

1. Failure of concrete in tension is clearly identified as resulting from the *creation and propagation of microcracks*. Damage variables, therefore, rely in this case on a strong physical basis. They can even be characterized experimentally [1].

2. Since the behaviour can, at least in a first approximation, be described as brittle elastic, the difficult problems arising from coupling damage with plasticity and hardening can be avoided.

The purpose of the present work is to lay down the basis for the description of elastic

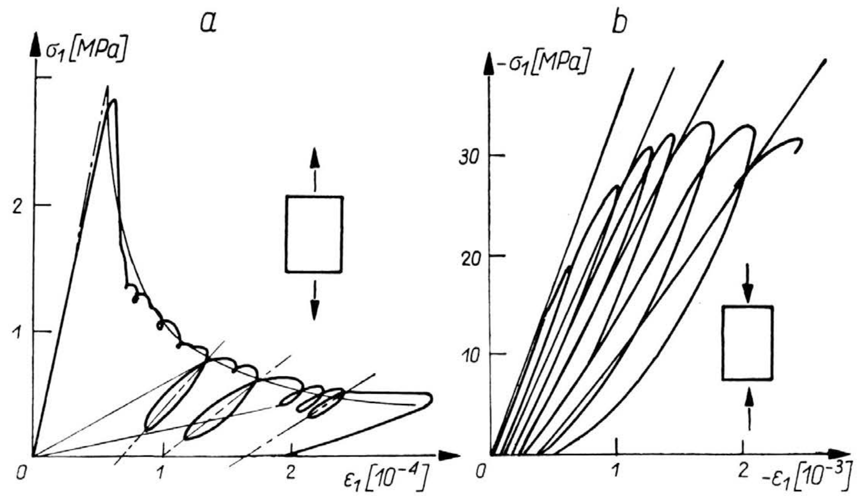



FIG. 1. a) TERRIEN, 1980, b) BENOUNICHE, 1979.

brittle materials with anisotropic damage and within the framework of generalized standard materials. A lot of models for concrete and rock mechanics have been proposed in the last few years [1–5], more or less explicitly related to damage concepts, but our objective is somewhat different and more restricted: of course these materials are the natural application of the model developed here but their behaviour will be used more as a guideline than as a final wished result. Indeed, it will be seen in the following that there remains *some fundamental problems to be solved before starting with the precise modelling and identification of a specific material.* It is precisely these problems that are investigated here. In particular, with respect to J. MAZARS's model [4] which was the starting point of our analysis, the essential points which will be investigated are:

a) taking into accunt anisotropic damage which is an essential feature about concrete [1].

b) postulating the damage evolution law as resulting from nonlinear irreversible thermodynamics.

Attention is restricted to infinitesimal deformations and standard notations will be used. In particular, the stress and strain tensors will be denoted by $\boldsymbol{\sigma}$ and $\boldsymbol{\epsilon}$. The deviatoric part of a tensor $\mathbf{a}$ is denoted by $\mathbf{a}^D$ and the scalar product of second-order tensors by $\mathbf{a}:\mathbf{b} = \mathrm{tr}\,\mathbf{a}\mathbf{b}^T$. Though we are interested in thermodynamics, thermomechanical coupling will be neglected and a purely mechanical theory is considered.

## 2. Elasticity coupled with damage

The damaged behaviour of brittle elastic material remains elastic. If $d$ denotes the damage variable and $A(d)$ and $\Lambda(d)$ the damaged stiffness and compliance tensors, the energy $W$ and the enthalpy $V$ are

$$W(\boldsymbol{\epsilon}, d) = \frac{1}{2}\boldsymbol{\epsilon}:A(d):\boldsymbol{\epsilon}, \qquad V(\boldsymbol{\sigma}, d) = \frac{1}{2}\boldsymbol{\sigma}:\Lambda(d):\boldsymbol{\sigma}. \tag{2.1}$$

There are many possible choices for the damage variable $d$ and for the damaged elasticity [6]. We shall use the approach previously developed in [7] and [8]: following Kachanov's ideas the damaged enthalpy is obtained by substitution of an effective stress tensor in the undamaged enthalpy:

$$V(\boldsymbol{\sigma}, d) = \frac{1}{2}\tilde{\boldsymbol{\sigma}}:\Lambda_0:\tilde{\boldsymbol{\sigma}}, \quad \tilde{\boldsymbol{\sigma}} = M(d):\boldsymbol{\sigma}, \tag{2.2}$$

where $M(d)$ is the fourth-order tensor giving the effective stress tensor $\tilde{\boldsymbol{\sigma}}$.

In order to describe anisotropic damage, the damage variable will be taken as a symmetric second-order tensor $\mathbf{D}$. For the sake of simplicity and to avoid unessential complications, we shall restrict ourselves to the triaxial case in which the principal axes of the stress and strain tensors are fixed so that $\mathbf{D}$ also has the same principal axes. It follows that the three special forms discussed in [7] for the effective stress tensor coincide and reduce to the simple form

$$\tilde{\boldsymbol{\sigma}} = \boldsymbol{\sigma}(\mathbf{1}-\mathbf{D})^{-1}, \quad \tilde{\sigma}_i = \frac{\sigma_i}{1-D_i}. \tag{2.3}$$

More generally we are interested in the framework which in [6] was called "uncoupled anisotropic damage". Substituting the form (2.3) in the undamaged enthalpy of an isotropic elastic material gives

$$V(\boldsymbol{\sigma}, \mathbf{D}) = \frac{1+\nu}{2E}\operatorname{tr}\boldsymbol{\sigma}^2(\mathbf{1}-\mathbf{D})^{-2} - \frac{\nu}{2E}[\operatorname{tr}\boldsymbol{\sigma}(\mathbf{1}-\mathbf{D})^{-1}]^2, \tag{2.4}$$

where $E$ and $\nu$, respectively, are Young's modulus and Poisson's ratio for the undamaged material. Constitutive equations give $\boldsymbol{\epsilon}$ and the thermodynamic force $\mathbf{G}$ associated to damage by

$$dV = \boldsymbol{\epsilon}:d\boldsymbol{\sigma}+\mathbf{G}:d\mathbf{D},$$

$$\begin{aligned} \boldsymbol{\epsilon} &= \frac{1+\nu}{E}\boldsymbol{\sigma}(\mathbf{1}-\mathbf{D})^{-2} - \frac{\nu}{E}[\operatorname{tr}\boldsymbol{\sigma}(\mathbf{1}-\mathbf{D})^{-1}](\mathbf{1}-\mathbf{D})^{-1}, \\ \mathbf{G} &= \frac{1+\nu}{E}\boldsymbol{\sigma}^2(\mathbf{1}-\mathbf{D})^{-3} - \frac{\nu}{E}[\operatorname{tr}\boldsymbol{\sigma}(\mathbf{1}-\mathbf{D})^{-1}]\boldsymbol{\sigma}(\mathbf{1}-\mathbf{D})^{-2}. \end{aligned} \tag{2.5}$$

As shown in [7] an alternative formulation could be developed starting from substitution of an effective strain in the undamaged energy:

$$W(\boldsymbol{\epsilon}, \mathbf{D}) = W_0(\tilde{\boldsymbol{\epsilon}}), \quad \tilde{\boldsymbol{\epsilon}} = \boldsymbol{\epsilon}(\mathbf{1}-\mathbf{D}),$$
$$dW = \boldsymbol{\sigma}:d\boldsymbol{\epsilon} - \mathbf{G}:d\mathbf{D} = \tilde{\boldsymbol{\sigma}}:d\tilde{\boldsymbol{\epsilon}},$$

while $\tilde{\boldsymbol{\sigma}}$ and $\tilde{\boldsymbol{\epsilon}}$ are related by the undamaged elastic law. In particular, the following expression of $\boldsymbol{\sigma}$ and $\mathbf{G}$ follows as a function of $\boldsymbol{\epsilon}$ and $\mathbf{D}$:

$$\begin{aligned} \boldsymbol{\sigma} &= \frac{E}{1+\nu}\left\{\boldsymbol{\epsilon}(\mathbf{1}-\mathbf{D})^2 + \frac{\nu}{1-2\nu}[\operatorname{tr}\boldsymbol{\epsilon}(\mathbf{1}-\mathbf{D})](\mathbf{1}-\mathbf{D})\right\}, \\ \mathbf{G} &= \frac{E}{1+\nu}\left\{\boldsymbol{\epsilon}^2(\mathbf{1}-\mathbf{D}) + \frac{\nu}{1-2\nu}[\operatorname{tr}\boldsymbol{\epsilon}(\mathbf{1}-\mathbf{D})]\boldsymbol{\epsilon}\right\}. \end{aligned} \tag{2.6}$$

For the uniaxial tension case, these relations reduce to

$$\tilde{\boldsymbol{\sigma}} = \begin{bmatrix} \frac{\sigma}{1-D_1} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \boldsymbol{\epsilon} = \begin{bmatrix} \varepsilon_1 & 0 & 0 \\ 0 & \varepsilon_2 & 0 \\ 0 & 0 & \varepsilon_3 \end{bmatrix}, \quad \mathbf{G} = \begin{bmatrix} G_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

(2.7)
$$\varepsilon_1 = \frac{\sigma}{\tilde{E}}, \quad \varepsilon_2 = -\frac{\tilde{\nu}\sigma}{\tilde{\sigma}}, \quad \tilde{E} = E(1-D_1)^2, \quad \tilde{\nu} = \nu\frac{1-D_2}{1-D_1},$$

$$G_1 = \frac{\sigma^2}{E(1-D_1)^3} = E\varepsilon^2(1-D_1).$$

## 3. Perfect damage

The damage evolution law will be obtained according to the general scheme of generalized standard material [9]: the damage rate $\dot{\mathbf{D}}$ will be taken as derivable from a dissipation potential $\omega$ depending on the thermodynamic force $\mathbf{G}$. Assuming rate independence, this potential $\omega$ will be the indicator function of an elastic domain defined from a damage threshold function:

(3.1)
$$\omega = 0 \quad \text{for} \quad p(\mathbf{G}) \leqslant 0,$$
$$\omega = \infty \quad \text{for} \quad p(\mathbf{G}) > 0,$$

which plays the same role as the yield function in plasticity. Since this threshold function does not change, this model will be called perfect damage, in analogy with perfect plasticity.

The evolution law gives $\dot{\mathbf{D}}$ as an element of the subdifferential of $\omega(\mathbf{G})$ which can be written as

(3.2)
$$\dot{\mathbf{D}} = 0 \qquad \text{if} \quad p(\mathbf{G}) < 0,$$
$$\dot{\mathbf{D}} = \mu\frac{\partial p}{\partial G}, \quad \mu > 0 \quad \text{if} \quad p(\mathbf{G}) = 0,$$

where the scalar quantity $\mu$ is determined by the condition that a damaging process must occur on the threshold surface:

(3.3)
$$\mu = -\left(\frac{\partial p}{\partial \mathbf{G}} : \frac{\partial \mathbf{G}}{\partial \mathbf{D}} : \frac{\partial p}{\partial \mathbf{G}}\right)^{-1} \left\langle \frac{\partial p}{\partial \mathbf{G}} : \frac{\partial \mathbf{G}}{\partial \boldsymbol{\epsilon}} : \dot{\boldsymbol{\epsilon}} \right\rangle$$

in which $\mathbf{G}$ is taken as a function of $\mathbf{D}$ and $\boldsymbol{\epsilon}$.

The simplest form is obtained by assuming this elastic domain to be a sphere in the $\mathbf{G}$ space

(3.4)
$$p(\mathbf{G}) = G_{II} - G_0 \leqslant 0, \quad \dot{\mathbf{D}} = \mu\frac{\mathbf{G}}{G_{II}}, \quad G_{II} = \sqrt{\mathbf{G}:\mathbf{G}},$$

where $\mathbf{G}_0$ is a material constant. This results in a brittle elastic model depending on three material constants $E$, $\nu$ and $G_0$.

For a uniaxial tension test, it follows from the relations (2.7) and (3.4) that

$$D_1 = D, \quad D_2 = D_3 = 0;$$

(3.5)
$$G_1 = G_0 = \frac{\sigma^2}{E(1-D)^3} = E\varepsilon^2(1-D),$$

which shows that the uniaxial tension curve $\sigma$, $\varepsilon$ is given by

$$\sigma = E\varepsilon, \quad D = 0 \quad \text{for} \quad \varepsilon \leqslant \varepsilon_0 = \sqrt{G_0/E}$$

(3.6)
$$\sigma = E\varepsilon_0\left(\frac{\varepsilon_0}{\varepsilon}\right)^3, \quad D = 1-\left(\frac{\varepsilon_0}{\varepsilon}\right)^2 \quad \text{for} \quad \varepsilon > \varepsilon_0,$$

which is represented in Fig. 2.

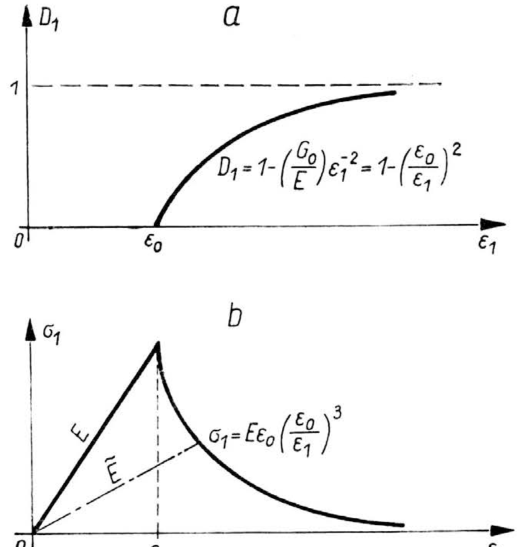



FIG. 2.

The material constant $G_0 = E\varepsilon_0^2$ can then be obtained from the elastic limit $\varepsilon_0$ in uniaxial tension.

## 4. Evoluting damage criterion

In the preceding model, the uniaxial tension curve ($\sigma$, $\varepsilon$) is given by the relations (3.6) and, except for the limit elastic strain $\varepsilon_0$, cannot be fitted to experimental data. This is obviously related to the fact that the damage criterion $p \leqslant 0$ remains fixed in the **G** space (perfect damage). A more general case will be obtained by assuming that this damage criterion also depends on a degradation variable $\beta$, which is analogous to the isotropic hardening variable in plasticity [8]. According to the general formalism of generalized standard materials, a "degradation" energy $\phi(\beta)$ is introduced in the energy or enthalpy. Therefore Eq. (2.4) is replaced by

$$V(\boldsymbol{\sigma}, \mathbf{D}, \beta) = V^e(\boldsymbol{\sigma}, \mathbf{D}) - \phi(\beta). \tag{4.1}$$

where $V^e(\boldsymbol{\sigma}, \mathbf{D})$ is the elastic enthalpy as given by Eq. (2.4). Introducing the thermodynamic force $B$ associated to $\beta$, the dissipation can be written as

$$\mathbf{G}:\dot{\mathbf{D}} - B\dot{\beta} \geqslant 0, \tag{4.2}$$

and the damage criterion and evolution laws are postulated as

$$p(\mathbf{G}, B) = G_{II} - B \leqslant 0, \quad B = B(\beta) = \frac{d\phi}{d\beta}, \tag{4.3}$$

$$\begin{aligned} \dot{\mathbf{D}} &= \mu \frac{\partial p}{\partial \mathbf{G}} = \mu \frac{\mathbf{G}}{G_{II}}, \\ \dot{\beta} &= -\mu \frac{\partial p}{\partial B} = \mu = \sqrt{\dot{\mathbf{D}}:\dot{\mathbf{D}}}\,. \end{aligned} \tag{4.4}$$

The damage limit $B$ now depends on a scalar variable which, from the relations (4.4), can be interpreted as a scalar cumulated equivalent damage. The scalar quantity $\mu$ can be written in a form similar to Eq. (3.3):

$$\mu = \left(\frac{dB}{d\beta} - \frac{dp}{d\mathbf{G}} : \frac{\partial \mathbf{G}}{\partial \mathbf{D}} : \frac{\partial p}{\partial \mathbf{G}}\right)^{-1} \left\langle \frac{\partial p}{\partial \mathbf{G}} : \frac{\partial \mathbf{G}}{\partial \boldsymbol{\epsilon}} : \dot{\boldsymbol{\epsilon}} \right\rangle. \tag{4.5}$$

Identification of the function $B(\beta)$ or $\phi(\beta)$ can be realized from a uniaxial test: in this case substituting the relations (2.7) in the evolution equation (4.4) and damage criterion (4.3), the following equations are obtained:

$$D_1 = D, \quad D_2 = D_3 = 0, \quad \beta = D,$$

$$E\varepsilon^2(1-D) = B(D),$$

which gives the parametric equation of the $(\sigma, \varepsilon)$ curve by

$$\sigma = E(1-D)^2\varepsilon, \quad \varepsilon = \sqrt{\frac{B(D)}{E(1-D)}}\,. \tag{4.6}$$

For example, if $B(\beta)$ is taken as

$$B(\beta) = G_0(1-B)^\alpha,$$

the $(\sigma, \varepsilon)$ relation can be written as

$$\sigma = E\varepsilon, \quad D = 0 \quad \text{for} \quad \varepsilon \leqslant \varepsilon_0 = \sqrt{G_0/E}\,,$$

$$\sigma = E\varepsilon_0 \left(\frac{\varepsilon_0}{\varepsilon}\right)^{\frac{3+\alpha}{1-\alpha}}, \quad D = \left(\frac{\varepsilon_0}{\varepsilon}\right)^{\frac{2}{1-\alpha}} \quad \text{for} \quad \varepsilon > \varepsilon_0,$$

and $\beta$ can be obtained from a power law identification of the $(\sigma, \varepsilon)$ -curve. More generally, the function $B(\beta)$ can be identified from the uniaxial tension curve.

## 5. Damage positiveness

It is well known that in the one-dimensional model the thermodynamic force $G$ can be interpreted as an elastic energy release rate [7] and that

$$G = E\varepsilon(1-D)^2 \geqslant 0, \quad \left(\frac{\partial G}{\partial D}\right)_\varepsilon = -2E\varepsilon(1-D) \leqslant 0,$$

which ensures the positiveness of the denominator in Eq. (3.3) and an always increasing damage according to the relations (3.2). Unfortunately, this is not so obviously true in the three-dimensional case. From the relations (2.6) the $G_1$ component of $\mathbf{G}$ is given by

$$\text{(5.1)} \qquad \frac{1+\nu}{E} G_1 = \frac{1-\nu}{1-2\nu} \psi_1 \varepsilon_1^2 + \frac{\nu}{1-2\nu} (\psi_2 \varepsilon_2 \varepsilon_1 + \psi_3 \varepsilon_3 \varepsilon_1),$$

where $\psi_i = 1-D_i > 0$. In most cases this will be positive and the predominant terms in $\mathbf{G}$ will be positive (for instance, it is easily shown that $\operatorname{tr}\mathbf{G} > 0$). But it may occur for some values of $\varepsilon_i$ and $D_i$ that one and perhaps two components of $G$ are negative. According to the relations (3.4) or (4.4) this would result in a decreasing value of the corresponding components of damage, which is not admissible.

In order to get rid of this difficulty, we replace in the damage criterion (3.4) the tensor $\mathbf{G}$ by its positive part, that is the tensor $\langle\mathbf{G}\rangle$ which has the same principal axes of $\mathbf{G}$ and the principal values of which are the positive parts of those of $\mathbf{G}$, $\langle G\rangle_i = \langle G_i\rangle$. We therefore replace for instance the relations (3.4) by:

$$\text{(5.2)} \qquad \begin{gathered} p(\mathbf{G}) = \hat{G}_{II} - G_0, \quad \hat{G}_{II} = \sqrt{\langle\mathbf{G}\rangle : \langle\mathbf{G}\rangle}, \\ \dot{\mathbf{D}} = \mu \frac{\partial G_{II}}{\partial \mathbf{G}} = \mu \frac{\langle\mathbf{G}\rangle}{\hat{G}_{II}}. \end{gathered}$$

In most cases, where all the components of $\mathbf{G}$ are positive, this results in no change for the model but in the other case this ensures an always increasing damage.

Extension of the second inequality (4.6) to the three-dimensional case is straightforward. Indeed, the differentiation (2.6) gives

$$-\frac{\partial p}{\partial \mathbf{G}} : \frac{\partial \mathbf{G}}{\partial \mathbf{D}} : \frac{\partial p}{\partial \mathbf{G}} = \frac{E}{1+\nu}\left\{\operatorname{tr}\boldsymbol{\epsilon}^2 \left(\frac{\partial p}{\partial \mathbf{G}}\right)^2 + \frac{\nu}{1-2\nu}\left(\operatorname{tr}\boldsymbol{\epsilon}\frac{\partial p}{\partial \mathbf{G}}\right)^2\right\},$$

which is obviously positive if $\partial p/\partial\mathbf{G}$ is positive definite as ensured earlier (at least in the triaxial case, there may be some difficulties in the general case). It follows that the incremental constitutive equations from the relations (3.3) and (4.5), provided $dB/d\beta$ is not too negative, are consistent. Of course, these incremental constitutive equations which can be constructed in terms of $\dot{\boldsymbol{\epsilon}}$ could not have been constructed in terms of $\boldsymbol{\sigma}$ because elastic brittle materials usually exhibit a decreasing stress-strain curve analogous to the softening case in plasticity [10].

## 6. Damage anisotropy index

Our model gives in uniaxial tension a "purely directional" damage $D_1$ ($D_2 = D_3 = 0$) which can be viewed as the development of microcracks perpendicular to the tension direction. On the other hand, an isotropic damage model with a scalar damage variable developed along the same line will lead to the same result (3.6) or (5.1) with $D_1 = D_2 = = D_3 = D$.

A damage anisotropy index $\delta$ was introduced in [7]. Expressing the degree of anisotropy of damage, it was defined as the ratio of transverse over longitudinal damage in uniaxial tension

$$D_1 = D, \quad D_2 = D_3 = \delta D. \tag{6.1}$$

Experimentally this coefficient could be obtained from a uniaxial tension test by measuring the damage stiffness $\tilde{E}$ and Poisson's ratio $\tilde{\nu}$.

At the present stage our model leads to $\delta = 0$ (anisotropic damage) or $\delta = 1$ (isotropic damage) but it cannot take into account an intermediate value $\delta_0$. In order to account for such a value, the following tensor is introduced:

$$\overline{\mathbf{G}} = \sin\theta_0 \frac{\operatorname{tr}\mathbf{G}}{\sqrt{3}} \mathbf{1} + \sqrt{\frac{3}{2}} \cos\theta_0 \mathbf{G}^D, \tag{6.2}$$

where $\theta_0$ is a fixed parameter. The damage criterion is then postulated as

$$\begin{gathered} p(\mathbf{G}) = \overline{G}_{II} - G_0 \leqslant 0, \\ \overline{G}_{II} = \sqrt{\overline{\mathbf{G}}:\overline{\mathbf{G}}} = \left(\sin^2\theta_0 (\operatorname{tr}\mathbf{G})^2 + \frac{3}{2}\cos^2\theta_0 \mathbf{G}^D:\mathbf{G}^D\right)^{1/2}, \end{gathered} \tag{6.3}$$

in the case of perfect damage.

The damage evolution law (3.2) then becomes

$$\dot{\mathbf{D}} = \mu \frac{\partial p}{\partial \mathbf{G}} = \frac{\mu}{G_{II}} \left(\sin^2\theta_0 (\operatorname{tr}\mathbf{G})\mathbf{1} + \frac{3}{2}\cos^2\theta_0 \mathbf{G}^D\right). \tag{6.4}$$

In the uniaxial tension case $\mathbf{G}$ is still given by the relations (2.7) so that Eqs. (6.3) and (6.4) reduce to

$$\begin{gathered} \overline{G}_{II} - G_0 \leqslant 0, \quad G_{II} = G_1 = E\varepsilon_1^2(1-D_1), \\ \dot{D}_1 = \mu, \quad \dot{D}_2 = \mu\left(\sin^2\theta_0 - \frac{1}{2}\cos^2\theta_0\right). \end{gathered} \tag{6.5}$$

The damage anisotropy index is therefore

$$\delta = \sin^2\theta_0 - \frac{1}{2}\cos^2\theta_0 = \frac{1}{2}(3\sin^2\theta_0 - 1). \tag{6.6}$$

Purely directional damage is obtained for $\delta = 0$, $\sin\theta_0 = 1/\sqrt{3}$ in which case $\overline{\mathbf{G}} = \mathbf{G}$. Isotropic damage is obtained for $\delta = 1$, $\theta_0 = \pi/2$. More generally, with this model the same $(\delta, \varepsilon)$-curve as before is obtained but $\theta_0$ can be identified to fit an experimental value of $\delta$.

## 7. Plane stress damage criterion

In order to illustrate the behaviour which results from these constitutive equations, plane stress situations will be considered:

$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{D} = \begin{bmatrix} D_1 & 0 & 0 \\ 0 & D_2 & 0 \\ 0 & 0 & D_3 \end{bmatrix}, \quad \mathbf{G} = \begin{bmatrix} G_1 & 0 & 0 \\ 0 & G_2 & 0 \\ 0 & 0 & 0 \end{bmatrix}; \tag{7.1}$$

$$G_1 = \frac{\sigma_1^2}{E(1-D_1)^3} - \frac{\nu}{E} \frac{\sigma_1 \sigma_2}{(1-D_2)(1-D_1)^2}$$

$$= \frac{E\varepsilon_1}{(1+\nu)(1-2\nu)} \{(1-\nu)\varepsilon_1(1-D_1)+\nu\varepsilon_2(1-D_2)\}$$

so that the damage criterion (4.3), (5.2) or (6.3) can be illustrated in the stress or strain plane. The initial damage criterion for $\mathbf{D} = 0$ is represented in Fig. 3 for some values of the anisotropy index $\delta$. Introduction of $\langle \mathbf{G} \rangle$ instead of $\mathbf{G}$ in order to ensure damage positiveness leads to small modifications of these criteria around the axes but they are too small to be noticed on these figures.

Stress strain relationships and damage evolution on some radial paths $\varepsilon_2 = \gamma\varepsilon_1$ are illustrated in Fig. 4 in the case of perfect damage for $\delta = 0$, that is in the model which was developed in Sect. 3. In particular, it should be noted that a radial path in the $\varepsilon_1 - \varepsilon_2$ plane does not lead to a radial path neither in the stress plane $(\sigma_1, \sigma_2)$ nor in the principal damage plane $(D_1, D_2)$.

## 8. Tension and compression

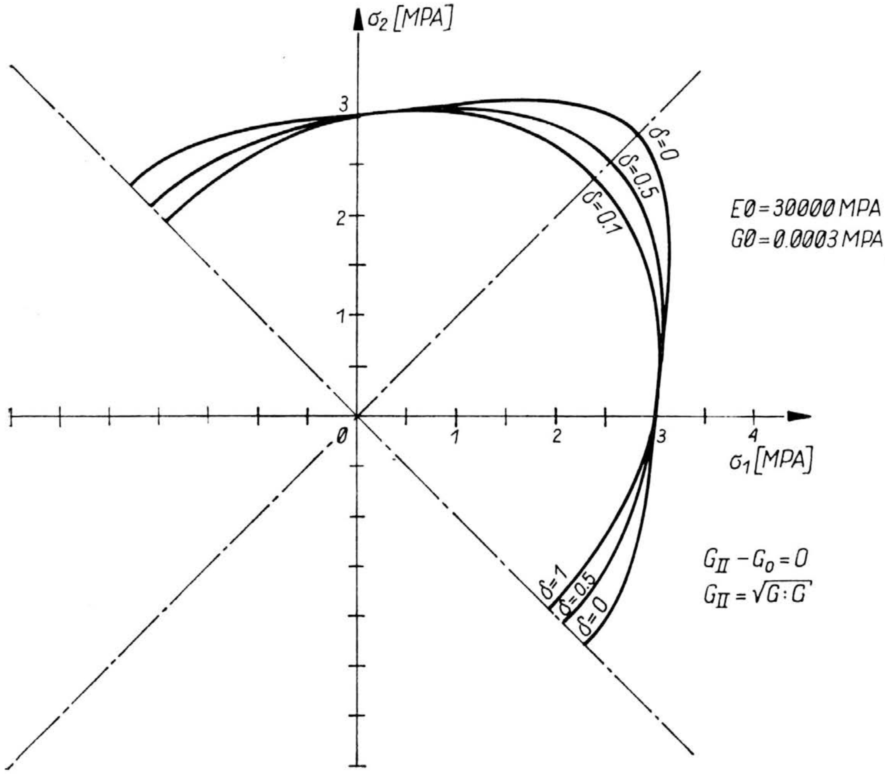
The model which has been presented here is appropriate for the description of brittle elastic materials like concrete and rocks in tension, but it cannot be applied for compression where it results in a behaviour which is entirely symmetric to tension behaviour. This results directly from the fact that the thermodynamic force $\mathbf{G}$ which is the fundamental variable is quadratic with respect to the stress tensor $\boldsymbol{\sigma}$ and is insensitive to its sign.

From the physical point of view this difference results from the contact problem on damage induced cracks. It is well known that the elastic response of a cracked body can be described by a hyperelastic nonquadratic elastic law [11] which, in the one-dimensional case, can be described by an elastic model with two moduli. This is due to cracks opening in tension and closing in compression. In other words, damage elastic coupling will be efficient in tension but it will disappear when the material is loaded in compression. In the one-dimensional case, this is easily achieved by replacing the usual definition of the effective stress $\tilde{\sigma}$ by

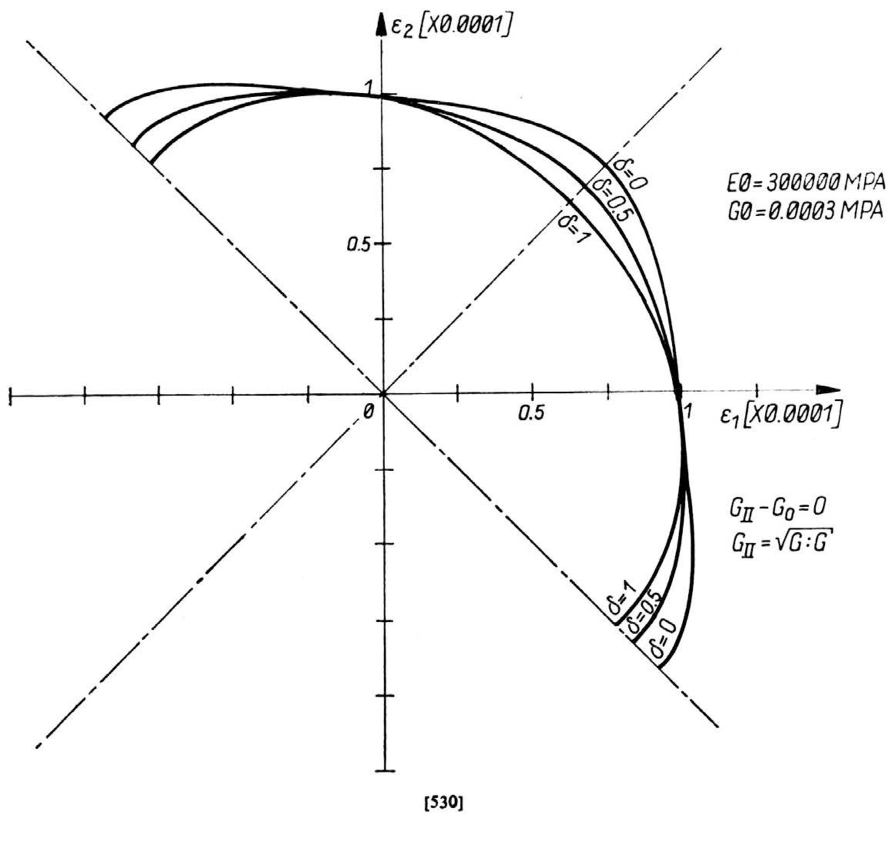
$$\tilde{\sigma} = \frac{\sigma}{1-\sigma} \quad \text{if} \quad \sigma > 0, \tag{8.1}$$

$$\tilde{\sigma} = \sigma \quad \text{if} \quad \sigma \leqslant 0.$$

The resulting model is brittle elastic in tension and elastic in compression. Further introduction of a classical plasticity model in compression will lead to a one-dimesional behaviour consistent with Fig. 1. The introduction of a second damage variable $d$ acting both in tension and compression will also allow for some damage in compression [12].



σ2 [MPA]
σ1 [MPA]
δ=0
δ=0.5
δ=0.1
δ=1
EØ=30000 MPA
GØ=0.0003 MPA
GII − G0 = 0
GII = √G:G

ε2 [X0.0001]
ε1 [X0.0001]
δ=0
δ=0.5
δ=1
EØ=300000 MPA
GØ=0.0003 MPA
GII − G0 = 0
GII = √G:G

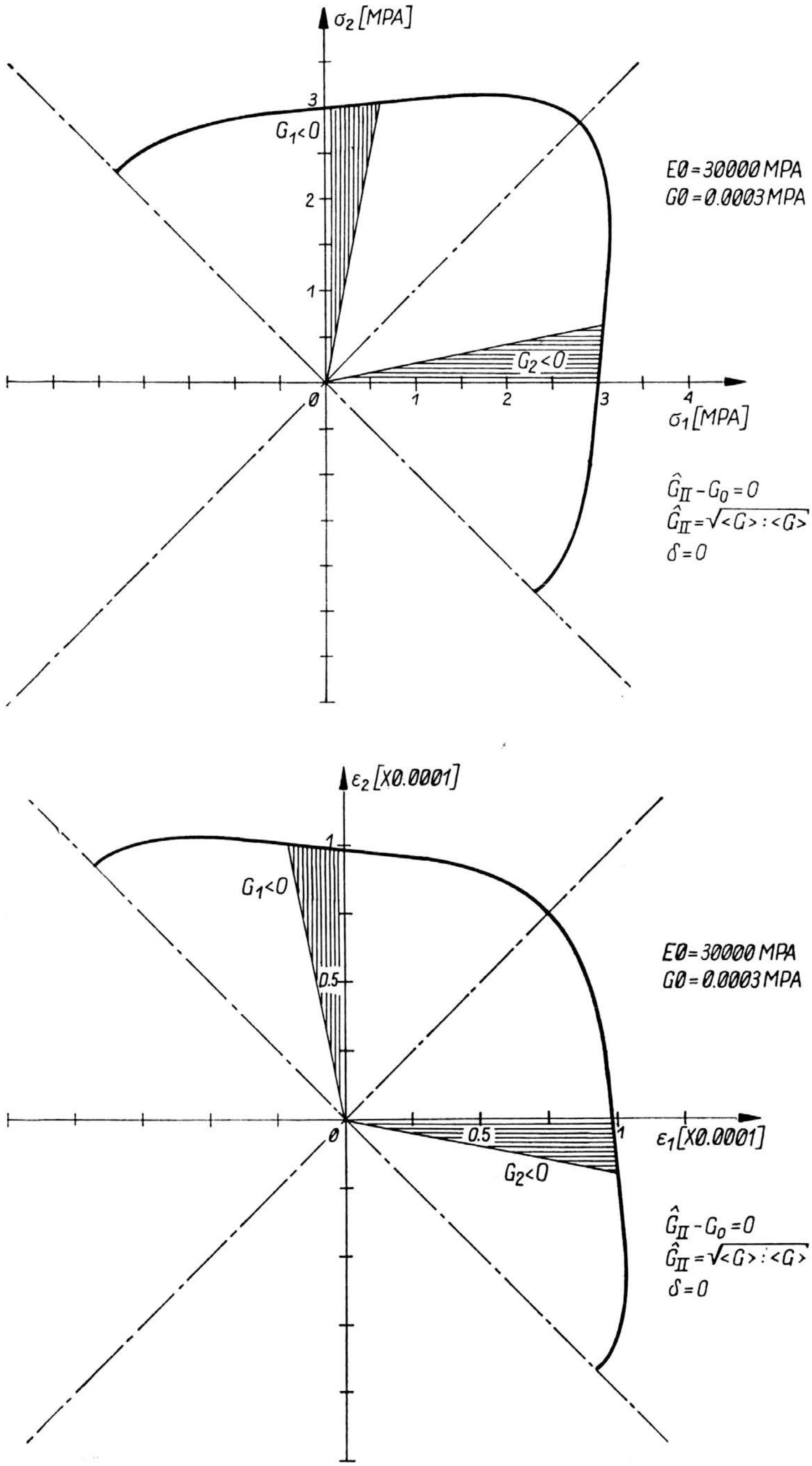

σ2 [MPA]
G1<0
G2<0
σ1 [MPA]
E0 = 30000 MPA
G0 = 0.0003 MPA
Ĝ_II − G_0 = 0
Ĝ_II = √<G>:<G>
δ = 0
ε2 [X0.0001]
G1<0
G2<0
ε1 [X0.0001]
E0 = 30000 MPA
G0 = 0.0003 MPA
Ĝ_II − G_0 = 0
Ĝ_II = √<G>:<G>
δ = 0

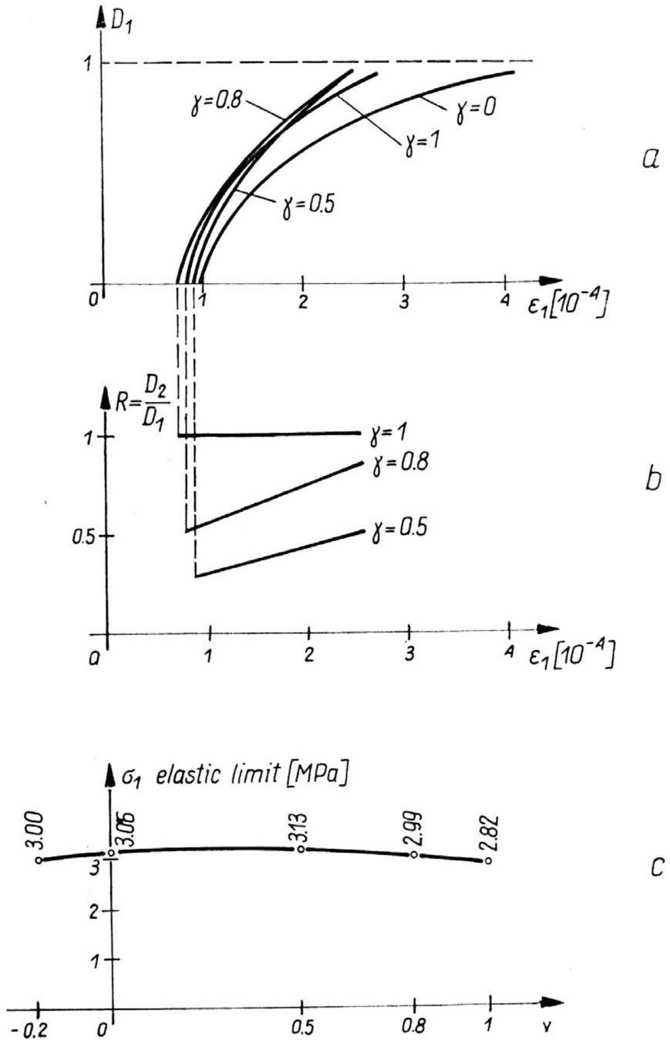
FIG. 3.

Compression damage may also induce friction along the generated cracks and this will result at the macroscopic level into some kind of plasticity [11] which could be taken into account along the line presented in [13].

The three-dimensional case is much more difficult to handle. Definition (8.1) can easily be extended into

$$
\begin{aligned}
\tilde{\sigma}_i &= \frac{\sigma_i}{1-D_i} && \text{if} \quad \sigma_i > 0 \\
\tilde{\sigma}_i &= \sigma_i && \text{if} \quad \sigma_i \leqslant 0.
\end{aligned}
\tag{8.2}
$$

Unfortunately, the resulting multielastic law is not admissible since it leads to a discontinuous response. There are some ways to overcome these difficulties which are now investigated [14] but the model will lose a great deal of its simplicity.



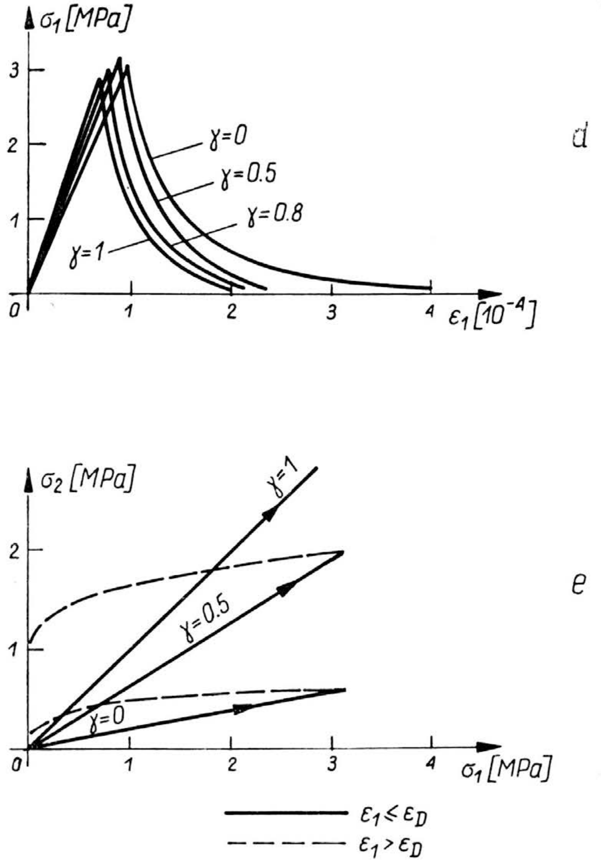



FIG. 4.

## 9. Conclusion

Some basic ideas about the mechanical and thermodynamical description of damage in brittle elastic materials have been presented. Many problems remain to be solved in order to obtain and identify a complete model for a specific material, like concrete for instance. The framework of generalized standard materials in the strict sense which has been used here will certainly prove to be too severe and it will be necessary to complete and modify it by some empirical of simplified assumptions, in particular by introducing in the damage evolution laws some other parameters than the thermodynamic force **G**. Homogenization results should also be used to give some more physical insight. However, the present work outlines the general framework which can be used.

## References

1. J. J. MARIGO, *Propagation des ondes ultrasonores et microfissuration du béton*, Thèse 3è Cycle, Paris VI, Janvier 1980.

2. A. DRAGON, Z. MRÓZ, *A continuum model for plastic brittle behaviour of rock and concrete*, Int. J. Engng. Sci., **17**, 1974.
3. D. KRAJCINOVIC, G. U. FONSEKA, *The continuous damage theory of brittle material*, Parts I, II, J. Appl. Mech., **48**, 809–815 and 816–824.
4. J. LEMAITRE, J. MAZARS, *Application de la théorie de l'endommagement au comportement non linéaire et à la rupture du béton de structure*, Annales de l'ITBTP, 401, 1982.
5. L. RESENDE, J. B. MARTIN, *A progressive damage continuum model for granular materials*, Comp. Meth. in Appl. Mech. Engin., **42**, 1–18, 1984.
6. F. SIDOROFF, *Description of anisotropic damage application to elasticity*, IUTAM Symposium, Senlis 1980.
7. J. P. CORDEBOIS, F. SIDOROFF, *Damage induced elastic anisotropy*, in: Mechanical Behaviour of Anisotropic Solids, J. P. BOEHLER [ed.], CNRS. Paris, 761–774, 1982.
8. J. P. CORDEBOIS, F. SIDOROFF, *Endommagement anisotrope en élasticité et plasticité*, J. de Méc. Théo. et Appl., Numéro Spécial, 1982.
9. B. HALPHEN, *Q. S.* NGUYEN, *Sur les matériaux standard généralisés*, J. Méca., **14**, 39–64, 1975.
10. F. SIDOROFF, *On the formulation of plasticity and viscoplasticity with internal variables*, Arch. Mech., **27**, 5–6, 807–819, 1975.
11. D. LEGUILLON, E. SANCHEZ-PALENCIA, *On the behaviour of a cracked elastic body with or without friction*, J. Mech. Theor. Appl., **1**, 1982.
12. F. SUPARTONO, F. SIDOROFF, *Modélisation de l'endommagement anisotrope du béton*, Congrès A.U.G.C. Cachan, Avril 1984.
13. J. F. CHENARD, F. SIDOROFF, *Sur un couplage endommagement–plasticité*, Congrès A.U.G.C. Cachan, Avril 1984.
14. F. SUPARTONO, *Contribution á la modelisation de l'endommagement anisotrope àu béton*, These, Ecole Centrale de Lyon, 1985.

LABORATOIRE DE MÉCANIQUE DES SOLIDES
ECOLE CENTRALE DE LYON, FRANCE.

*Received October* 8, 1984.