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Limit analysis theorems in the case of Signorini's boundary conditions and friction

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THE OBJECTIVE of this paper is to extend the lower and upper bound theorems of the classical limit analysis to Signorini's boundary conditions without and with friction. The friction condition can be nonconvex while the subdifferential friction law is not necessarily associated with this condition.

Celem pracy jest rozszerzenie twierdzeń o dolnej i górnej granicy obciążenia, stanowiących podstawę klasycznej teorii nośności granicznej, na przypadek warunków brzegowych Signoriniego bez tarcia i z tarciem. Warunek tarcia może być niewypukły, natomiast subróżniczkowe prawo tarcia nie musi być z tym warunkiem stowarzyszone.

Целью работы является расширение теорем о нижнем и верхнем пределах нагрузки, составляющих основу классической теории предельной несущей способности, на случай граничных условий Синьорини без трения и с трением. Условие трения может быт невыпуклым, субдифференциальный закон трения же не должен быть ассоциирован с этим условием.

1. Introduction

THE "CLASSICAL" limit analysis deals with a perfectly-plastic body subjected to a one-parameter loading [15, 26, 27], while boundary conditions are of the bilateral type. Multi-parameter loadings can also be considered [11, 21–23, 29]. It seems that even the classical theory is not satisfactorily formulated since in the presence of body forces the load multiplier affects both a surface traction and body forces, see Refs. [15, 26, 31].

The statical and kinematical methods of multi-parameter limit analysis have been generalized to a class of unilateral frictionless constraints by FRÉMOND [10]. COLLINS [4] has formulated the upper bound technique of an estimation of a total load acting on a part of the boundary. This approach, which incorporates Coulomb's friction, is valid for a restricted class of boundary value problems in the absence of Signorini's conditions.

Contact problems usually belong to the so-called free surface problems, even in the case of a contact of an elastic body with a rigid support. The modern approach to such problems consists in using the methods of convex analysis, variational inequalities and implicit variational inequalities. A comprehensive survey of applications of variational methods to various contact problems for solids and structures is given in the paper [30].

The present work is concerned with a generalization of the lower and upper bound theorems of the classical one-parameter limit analysis to a larger class of boundary conditions than that usually studied in the relevant literature. The situations which will be dealt with can be described as follows. Let us assume that the boundary Γ of a rigid, perfectly-plastic body consists of three parts: Γ_0, Γ_1 and Γ_2. On Γ_0 and Γ_1 the classical bilateral boundary conditions are prescribed. On the other hand two types of boundary conditions, imposed on Γ_2, will be considered: 1) frictionless Signorini's conditions, and 2) Signorini's conditions with not necessarily Coulomb's friction. Associated and nonassociated friction laws are also briefly discussed. Nonlocal friction laws introduced by DUVAUT [7] will not be studied in this paper. Other types of boundary conditions imposed on Γ_2 as well as various unilateral boundary conditions for locking bodies [5, 6] will be studied separately.

2. Limit analysis theorems in the presence of Signorini's boundary conditions without friction

Let a rigid, perfectly-plastic body occupy a sufficiently regular region of three-dimensional Euclidean space R^3. Throughout this paper the boundary Γ of Ω will always consists of three disjoint parts: $\Gamma_0, \Gamma_1, \Gamma_2$, such that $\Gamma = \bar{\Gamma}_0 \cup \bar{\Gamma}_1 \cup \bar{\Gamma}_2$ and the surface measure of Γ_0 is positive. Here the bar over a set denotes its closure. We shall always assume that on Γ_0 and Γ_1 the "classical" conditions are imposed. Γ_2 is the surface of a possible contact. In this section we shall formulate and prove the limit analysis theorems for frictionless Signorini's boundary conditions imposed on Γ_2.

Let $\mathbf{n} = (n_i)$ denote a unit exterior normal vector to Γ. Throughout this paper Latin indices run from 1 to 3. A vector $\mathbf{v} = (v_i)$ defined on Γ can be decomposed as follows:

$$\mathbf{v} = v_N \mathbf{n} + \mathbf{v}_T, \tag{2.1}$$

where $v_N = v_i n_i$ denotes the normal component of $\mathbf{v}$, while $v_{Ti} = v_i - v_N n_i$ stands for tangential components.

Similar decomposition can be carried out for a stress vector $(\tau_{ij} n_j)$ if $\boldsymbol{\tau} = (\tau_{ij})$ is a stress tensor. Thus we have

$$\tau_{ij} n_j = \tau_N n_i + \tau_{Ti}, \tag{2.2}$$

where $\tau_N = \tau_{ij} n_i n_j$, $\tau_{Ti} = \tau_{ij} n_j - \tau_N n_i$.

Let $\mathbf{u} = (u)$ and $\boldsymbol{\sigma} = (\sigma_{ij})$ denote a velocity field and a stress field, respectively, solving the problem of limit analysis to be made precise below. Signorini's unilateral boundary conditions, in the absence of friction, are given by

$$\begin{aligned} &\boldsymbol{\sigma}_T = 0, \\ &u_N \leqslant 0, \quad \sigma_N \leqslant 0, \quad \sigma_N u_N = 0 \end{aligned} \quad \text{on } \Gamma_2. \tag{2.3}$$

If C is a convex of plasticity, then the indicator function χ_C is defined as follows [19]:

$$\chi_C(\boldsymbol{\tau}) = \begin{cases} 0, & \text{if} \quad \boldsymbol{\tau} \in C, \\ +\infty, & \text{if} \quad \boldsymbol{\tau} \notin C. \end{cases} \tag{2.4}$$

For instance, if $f(\boldsymbol{\tau}) = 0$ defines the yield surface, then

$$C (\{\boldsymbol{\tau} = (\boldsymbol{\tau}_{ij}) | \tau_{ij} = \tau_{ji},\ f(\boldsymbol{\tau}) \leqslant 0\}. \tag{2.5}$$

We observe that in this paper we do not make a difference, from the point of view of notations, between a field and its value at a point. Such ambiguity is allowable if one remains at the physical level of accuracy.

The support function d of C is given by (¹)

$$(2.6) \qquad d(\boldsymbol{\epsilon}) = \sup_{\boldsymbol{\tau}} \left(\boldsymbol{\tau}\cdot\boldsymbol{\epsilon} - \chi_C(\boldsymbol{\tau})\right) = \sup_{\boldsymbol{\tau}\in C} (\boldsymbol{\tau}\cdot\boldsymbol{\epsilon})$$

where $\boldsymbol{\tau}\cdot\boldsymbol{\epsilon} = \tau_{ij}\varepsilon_{ij}$ and $\boldsymbol{\epsilon}$ is the strain rate tensor. The associated flow rule can be written in the subdifferential form

$$(2.7) \qquad \boldsymbol{\epsilon} \in \partial\chi_C(\boldsymbol{\tau}), \quad \text{or} \quad \boldsymbol{\tau} \in \partial d(\boldsymbol{\epsilon})$$

where $\partial d(\boldsymbol{\epsilon})$ is the subdifferential of d at $\boldsymbol{\epsilon}$. Each of the conditions (2.7) is equivalent to

$$(2.8) \qquad \chi_C(\boldsymbol{\tau}) + d(\boldsymbol{\epsilon}) = \boldsymbol{\tau}\cdot\boldsymbol{\epsilon}.$$

Since $\boldsymbol{\tau} \in C$, then Eq. (2.8) results in

$$(2.9) \qquad d(\boldsymbol{\epsilon}) = \boldsymbol{\tau}\cdot\boldsymbol{\epsilon}.$$

Hence we infer that d is the density of the plastic dissipation. Convexity implies that d is non-negative provided that the stress and strain rate tensors are interrelated by the flow rule (2.7). The total dissipation is given by

$$(2.10) \qquad D(\boldsymbol{\epsilon}) = \int_{\Omega} d\left(\boldsymbol{\epsilon}(\mathbf{x})\right) d\mathbf{x}.$$

The volume integral is assumed to include contributions from velocity discontinuities. It can be assumed that D is convex and subdifferentiable, cf. Refs. [21–23].

A stress field $\boldsymbol{\tau}$ defined over Ω is said to be *statically admissible* if

$$(2.11) \qquad \tau_{ij,j} + b_i = 0, \quad \text{in} \quad \Omega,$$

$$(2.12) \qquad \boldsymbol{\tau}(\mathbf{x}) \in C(\mathbf{x}), \quad \text{in} \quad \Omega,$$

$$(2.13) \qquad \tau_{ij}n_j = \mu^s p_i^0, \quad \text{on} \quad \Gamma_1,$$

$$(2.14) \qquad \boldsymbol{\tau}_T = 0, \quad \boldsymbol{\tau}_N \leqslant 0, \quad \text{on} \quad \Gamma_2.$$

Here μ^s is the static load multiplier, whereas $\mathbf{p}^0 = (p_i^0)$ is prescribed. We observe that μ^s does not affect body forces $\mathbf{b}$. Such an approach seems to me reasonable since the usually considered „body forces" $\mu^s\mathbf{b}$ are physically unrealizable within the framework of limit analysis.

A velocity field $\mathbf{v}$ defined over Ω is said to be *kinematically admissible* if

$$(2.15) \qquad \mathbf{v} = 0, \quad \text{on} \quad \Gamma_0,$$

$$(2.16) \qquad v_N \leqslant 0, \quad \text{on} \quad \Gamma_2,$$

$$(2.17) \qquad \int_{\Gamma_1} \mathbf{p}^0 \cdot \mathbf{v}\, d\Gamma > 0.$$

The condition (2.17) is discussed by BORKOWSKI [2]. For incompressible materials such a field must additionally satisfy the incompressibility condition

$$(2.18) \qquad v_{i,i} = 0, \quad \text{in} \quad \Omega.$$

(¹) Notions of convex analysis used in this paper can be found in the monograph by ROCKAFELLAR [19]

The kinematically admissible load multiplier μ^k is defined as follows:

$$\mu^k(\mathbf{v}) = \frac{D(\boldsymbol{\epsilon}(\mathbf{v})) - \int\limits_{\Omega} \mathbf{b}\cdot\mathbf{v}d\mathbf{x}}{\int\limits_{\Gamma_1} \mathbf{p}^0\cdot\mathbf{v}d\Gamma}, \tag{2.19}$$

where $\varepsilon_{ij}(\mathbf{v}) = (v_{i,j}+v_{j,i})/2$. If the load multiplier affects also body forces, then instead of Eq. (2.19) we have [15, 27]

$$\tilde{\mu}^k(\mathbf{v}) = \frac{D(\boldsymbol{\epsilon}(\mathbf{v}))}{\int\limits_{\Gamma_1} \mathbf{p}^0\cdot\mathbf{v}d\Gamma + \int\limits_{\Omega} \mathbf{b}\cdot\mathbf{v}d\mathbf{x}}. \tag{2.20}$$

The difference between the definitions (2.19) and (2.20) of the kinematic load multiplier is obvious.

By a *complete solution* of the limit analysis problem we mean a triple $(\mu, \boldsymbol{\sigma}, \mathbf{u})$ such that the stress field $\boldsymbol{\sigma}$ is statically admissible, the velocity field $\mathbf{u}$ is kinematically admissible, whereas μ is the associated load multiplier, that is $\mu = \mu^s(\boldsymbol{\sigma}) = \mu^k(\mathbf{v})$. Moreover, $\boldsymbol{\sigma}$ and $\boldsymbol{\epsilon}(\mathbf{u})$ are interrelated by the flow rule $(2.7)_1$ or $(2.7)_2$.

2.1. Lower bound theorem

We shall prove that

$$\mu^s(\boldsymbol{\tau}) \leqslant \mu. \tag{2.21}$$

In other words the lower bound theorem reads

$$\left|\begin{array}{l}\text{find}\\ \qquad \sup\limits_{\boldsymbol{\tau}\in K_s} \mu^s(\boldsymbol{\tau}),\end{array}\right. \tag{2.22}$$

where

$$K_s = \{\boldsymbol{\tau} = (\tau_{ij}) | \boldsymbol{\tau}(\mathbf{x}) \in C(\mathbf{x}),\ \mathbf{x}\in\Omega,\ \tau_{ij,j}+b_i = 0;\ \tau_{ij}n_j = \mu^s p_i^0,\ \Gamma_1;\ \tau_N \leqslant 0,\ \boldsymbol{\tau}_T = 0,\ \Gamma_2\} \tag{2.23}$$

is closed and convex in an appropriately chosen space, cf. Refs. [31, 32].

To prove the inequality (2.21) we consider the equality

$$\int\limits_{\Omega} (\tau_{ij}-\sigma_{ij})_{,j} u_i d\mathbf{x} = 0, \qquad \forall\boldsymbol{\tau}\in K_s. \tag{2.24}$$

Integrating the last equation by parts, carrying out decompositions according to the relations (2.1) and (2.2) and taking account of the equality $\sigma_N u_N = 0$, we readily arrive at

$$0 \leqslant \int\limits_{\Omega} (\tau_{ij}-\sigma_{ij})\,\varepsilon_{ij}(\mathbf{u})d\mathbf{x} = (\mu^s-\mu)\int\limits_{\Gamma_1} p_i^0 u_i d\Gamma + \int\limits_{\Gamma_2} \tau_N u_N d\Gamma, \qquad \forall\boldsymbol{\tau}\in K_s. \tag{2.25}$$

The inequalities $\tau_N \leqslant 0$ and $u_N \leqslant 0$ imply $\tau_N u_N \geqslant 0$. Hence we eventually obtain the relation (2.21).

2.2. Upper bound theorem

In the present subsection we shall prove the inequality

$$\mu \leqslant \mu^k(\mathbf{v}). \tag{2.26}$$

Thus the kinematic approach is equivalent to the following minimization problem:

$$\left| \begin{array}{l} \text{find} \\ \quad \inf\limits_{\mathbf{v}\in K_v} \mu^k(\mathbf{v}), \end{array} \right. \tag{2.27}$$

where

$$K_v = \{\mathbf{v} = (v_i) | \mathbf{v} = 0,\ \Gamma_0;\ v_N \leqslant 0, \Gamma_2\} \tag{2.28}$$

or, in the case of incompressible materials,

$$K_v = \{\mathbf{v} = (v_i) | v_{i,i} = 0, \Omega;\ \mathbf{v} = 0,\ \Gamma_0;\ v_N \leqslant 0, \Gamma_2\}. \tag{2.29}$$

The subdifferentiability of the functional D means that

$$D(\boldsymbol{\epsilon}(\mathbf{v})) - D(\boldsymbol{\epsilon}(\mathbf{u})) \geqslant \int\limits_{\Omega} \sigma_{ij}\varepsilon_{ij}(\mathbf{v}-\mathbf{u})dx, \qquad \forall \mathbf{v} \in K_v. \tag{2.30}$$

In conformity with Eq. (2.19), we substitute

$$\mu^k \int\limits_{\Gamma_1} \mathbf{p}^0 \cdot \mathbf{v} d\Gamma + \int\limits_{\Omega} \mathbf{b} \cdot \mathbf{v} d\mathbf{x}, \quad \mu \int\limits_{\Gamma_1} \mathbf{p}^0 \cdot \mathbf{u} d\Gamma + \int\limits_{\Omega} \mathbf{b} \cdot \mathbf{u} d\mathbf{x},$$

for $D(\boldsymbol{\epsilon}(\mathbf{v}))$ and $D(\boldsymbol{\epsilon}(\mathbf{u}))$, respectively. Thus we readily arrive at

$$\mu^k \int\limits_{\Gamma_1} \mathbf{p}^0 \cdot \mathbf{v} d\Gamma \geqslant \mu \int\limits_{\Gamma_1} \mathbf{p}^0 \cdot \mathbf{v} d\Gamma + \int\limits_{\Gamma_2} \sigma_N v_N d\Gamma. \tag{2.31}$$

Since $\sigma_N v_N \geqslant 0$ and the relation (2.17) holds, we eventually obtain the inequality (2.26).

REMARK 2.1. Signorini's condition $\sigma_N u_N = 0$ is a natural consequence of the so-called extremality relations [1, 31, 32] provided that the optimization problems (2.22) and (2.27) are handled from the point of view of duality. This problem will be studied separately.

REMARK 2.2. An examination of the problem of uniqueness of a complete solution $(\mu, \boldsymbol{\sigma}, \mathbf{u})$ results in conclusions similar to the case of bilateral boundary conditions, therefore it is omitted here.

REMARK 2.3. Upper and lower bound theorems will remain valid if, instead of the relation (2.19), the following definition of the kinematic load multiplier is assumed:

$$\mu_1^k(\mathbf{v}) = \frac{\int\limits_{\Omega} (\mathbf{t} \cdot \boldsymbol{\epsilon}(\mathbf{v}) - \mathbf{b} \cdot \mathbf{v}) d\mathbf{x} - \int\limits_{\Gamma_2} t_N v_N d\Gamma}{\int\limits_{\Gamma_1} \mathbf{p}^0 \cdot \mathbf{v} d\Gamma}, \tag{2.32}$$

where $\mathbf{t} \in \partial d(\boldsymbol{\epsilon}(\mathbf{v}))$. The formula (2.32) results directly from the principle of virtual velocities. We observe that $\mu^k(\mathbf{u}) = \mu_1^k(\mathbf{u})$, provided that $\mathbf{u}$ enters a complete solution, since then $\sigma_N u_N = 0$.

REMARK 2.4. The lower and upper bound theorems established in this section hold with minor and evident modifications for the case when Γ_2 is an interface, that is $\Omega = \Omega_1 \cup \Omega_2 \cup \Gamma_2$. Signorini's conditions take on the form

$$(2.33) \qquad \boldsymbol{\sigma}_T = \boldsymbol{\sigma}_T^{(1)} = -\boldsymbol{\sigma}_T^{(2)} = 0, \quad [\![u_N]\!] \leqslant 0, \quad \sigma_N \leqslant 0, \quad [\![u_N]\!]\sigma_N = 0, \quad \text{on } \Gamma_2$$

where $\sigma_N = \sigma_N^{(1)} = -\sigma_N^{(2)}$, $[\![u_N]\!]$ stands for the jump of the normal component of the velocity field across Γ_2. The superscripts (1), (2) refer to Ω_1, Ω_2, respectively. On Γ_2 the unit normal $\mathbf{n}$ is taken as exterior to Ω_1 and $[\![u_N]\!] = u_N^{(1)} - u_N^{(2)}$.

3. Associated and nonassociated friction laws

It is well known that Coulomb's friction condition is given by

$$(3.1) \qquad |\boldsymbol{\tau}_T| - \nu|\tau_N| \leqslant 0,$$

where $|\boldsymbol{\tau}_T| = \sqrt{\tau_{Ti}\tau_{Ti}}$, $\nu = \nu(\mathbf{x})$ is the coefficient of friction; here $\mathbf{x} \in \Gamma_2$. The friction condition (3.1) is isotropic and the set

$$(3.2) \qquad K = \{\boldsymbol{\tau}_T : |\boldsymbol{\tau}_T| - \nu|\tau_N| \leqslant 0, \quad \text{on} \quad \Gamma_2\}$$

is closed and convex.

In the general case a friction condition will be given by

$$(3.3) \qquad g(\tau_N, \boldsymbol{\tau}_T) \leqslant 0.$$

Some restrictions which must be imposed on g will be delivered below. However, it is not necessary to assume the convexity of the set

$$(3.4) \qquad K = \{\boldsymbol{\tau} | g(\tau_N, \boldsymbol{\tau}_T) \leqslant 0\}.$$

The function g can be anisotropic, for instance orthotropic.

If τ_N is fixed, then we set

$$(3.5) \qquad h(\boldsymbol{\tau}_T) = g(\tau_N, \boldsymbol{\tau}_T).$$

Coulomb's friction law is expressed as follows:

$$(3.6) \qquad \begin{aligned} &|\boldsymbol{\tau}_T| < \nu|\tau_N| \Rightarrow \mathbf{v}_T = 0, \\ &|\boldsymbol{\tau}_T| = \nu|\tau_N| \Rightarrow \mathbf{v}_T = -\lambda\boldsymbol{\tau}_T, \qquad \lambda \geqslant 0. \end{aligned}$$

Hence we infer that if h is obtained from Coulomb's friction condition, then

$$(3.7) \qquad \mathbf{v}_T = -\tilde{\lambda}\frac{\partial h}{\partial \boldsymbol{\tau}_T} = -\lambda\boldsymbol{\tau}_T,$$

where $\lambda = \tilde{\lambda}/|\boldsymbol{\tau}_T|$. Thus we conclude that the friction law (3.6) is not associated with Coulomb's friction condition (3.1).

If $\tau_N = F_N$, where F_N is prescribed on Γ_2, then we have to do with the associated flow rule.

Generally we assume that the set

$$(3.8) \qquad K(\sigma_N) = \{\boldsymbol{\tau}_T | g(\sigma_N, \boldsymbol{\tau}_T) \leqslant 0\}$$

is closed and convex for each fixed σ_N satisfying the inequality (3.3). Thus, as we previously mentioned, the nonconvex sets K defined by the relation (3.4) are admissible. Nonconvex friction conditions were obtained in [17] when studying asperities from the geometrical point of view. Nonconvex yield conditions are reported by SALENÇON and TRISTÁN-LÓPEZ [25].

The general law of friction is here assumed in the form

$$-\mathbf{u}_T \in \partial\chi_{K(\sigma_N)}(\boldsymbol{\sigma}_T), \tag{3.9}$$

where

$$\chi_{K(\sigma_N)}(\boldsymbol{\tau}_T) = \begin{cases} 0, & \text{if} \quad \boldsymbol{\tau}_T \in K(\sigma_N), \\ +\infty, & \text{if} \quad \boldsymbol{\tau}_T \notin K(\sigma_N). \end{cases} \tag{3.10}$$

The subdifferential friction law (3.9) is equivalent to

$$(\boldsymbol{\tau}_T - \boldsymbol{\sigma}_T)\cdot\mathbf{u}_T \geqslant 0 \quad \forall \boldsymbol{\tau}_T \in K(\sigma_N). \tag{3.11}$$

Denoting by $\boldsymbol{\tau}_T^*$ the conjugate variable, the support function $\psi(\sigma_N, \boldsymbol{\tau}_T^*)$ of the convex set $K(\sigma_N)$ is given by

$$\psi(\sigma_N, \boldsymbol{\tau}_T^*) = \sup_{\boldsymbol{\tau}_T} \left(\boldsymbol{\tau}_T^*\cdot\boldsymbol{\tau}_T - \chi_{K(\sigma_N)}(\boldsymbol{\tau}_T)\right) = \sup_{\boldsymbol{\tau}_T \in K(\sigma_N)} \left(\boldsymbol{\tau}_T^*\cdot\boldsymbol{\tau}_T\right). \tag{3.12}$$

The function $\psi(\sigma_N, \cdot)$ is positively homogeneous of degree one [19].

Duality of the indicator function $\chi_{K(\sigma_N)}(\cdot)$ with the support function $\psi(\sigma_N, \cdot)$ results in

$$\boldsymbol{\sigma}_T \in \partial_2\psi(\sigma_N, -\mathbf{u}_T), \tag{3.13}$$

where $\partial_2\psi(\sigma_N, -\mathbf{u}_T)$ denotes the subdifferential of the function $\psi(\sigma_N, \cdot)$. Hence we infer that $\boldsymbol{\sigma}_T^* = -\mathbf{u}_T$. Either of the solutions (3.9) or (3.13) is equivalent to

$$\chi_{K(\sigma_N)}(\boldsymbol{\sigma}_T) + \psi(\sigma_N, -\mathbf{u}_T) = \boldsymbol{\sigma}_T\cdot(-\mathbf{u}_T), \tag{3.14}$$

or, since $\boldsymbol{\sigma}_T \in K(\sigma_N)$,

$$j(\sigma_N, \mathbf{u}_T) \overset{\text{df}}{=} \psi(\sigma_N, -\mathbf{u}_T) = \boldsymbol{\sigma}_T\cdot(-\mathbf{u}_T). \tag{3.15}$$

The function $j(\sigma_N, \mathbf{u}_T)$ represents the dissipation density of friction stresses $\boldsymbol{\sigma}_T$. Therefore we call it the friction dissipation density. We observe that due to the nonassociated character of the friction law (3.9) or (3.13), the friction dissipation depends explicitly on normal stresses and not only on the tangential components of a velocity field. The function j is convex with respect to the second argument, but not necessarily jointly convex.

EXAMPLE 3.1. Suppose that g is symmetric (isotropic) in the following sense:

$$g(A_1\tau_N, \mathbf{A}_2\boldsymbol{\tau}_T) = g(\tau_N, \boldsymbol{\tau}_T), \tag{3.16}$$

where A_1 is the orthogonal linear transformation which reverses the sign of the normal component, whereas $\mathbf{A}_2$ is an orthogonal transformation. Then we have, see Ref. [19],

$$g(\tau_N, \boldsymbol{\tau}_T) = G(|\tau_N|, |\boldsymbol{\tau}_T|). \tag{3.17}$$

Hence we obtain

$$j(\sigma_N, \mathbf{u}_T) = |\boldsymbol{\sigma}_T||\mathbf{u}_T|. \tag{3.18}$$

In the specific case of Coulomb's friction the last relation immediately yields

$$j(\sigma_N, \mathbf{u}_T) = \nu|\sigma_N||\mathbf{u}_T|. \tag{3.19}$$

EXAMPLE 3.2. Let us assume that Γ_2 is a regular domain in the (x_1, x_2)-plane. Then we have $\boldsymbol{\tau}_T = (\tau_{T\alpha}, 0)$, $\mathbf{u}_T = (u_{T\alpha}, 0)$ $\alpha = 1, 2$. It is natural to consider the following anisotropic friction condition:

$$g(\tau_N, \boldsymbol{\tau}_T) = \frac{1}{2}N_{\alpha\beta}\tau_{T\alpha}\tau_{T\beta} - H(\tau_N), \quad \text{on} \quad \Gamma_2. \tag{3.20}$$

The tensor $\mathbf{N}$ can be called the anisotropic friction tensor, cf. [33]. We assume that $N_{\alpha\beta} = N_{\beta\alpha}$, rank $\mathbf{N} = 2$. Now we have

$$K(\sigma_N) = \left\{ \boldsymbol{\tau}_T = (\tau_{T1}, \tau_{T2}) \,\middle|\, \frac{1}{2} N_{\alpha\beta} \tau_{T\alpha} \tau_{T\beta} - H(\sigma_N) \leqslant 0 \right\}. \tag{3.21}$$

The set $K(\sigma_N)$ is convex if and only if $\mathbf{N}$ is positive definite.

The non-associated friction law takes on the form

$$u_{T\alpha} = -\lambda N_{\alpha\beta} \sigma_{T\beta}, \quad \lambda \geqslant 0, \tag{3.22}$$

where

$$\lambda = \frac{(M_{\alpha\beta} u_{T\alpha} u_{T\beta})^{1/2}}{(2H(\sigma_N))^{1/2}}, \quad \mathbf{M} = \mathbf{N}^{-1}. \tag{3.23}$$

The friction dissipation density is expressed as follows:

$$j(\sigma_N, \mathbf{u}_T) = \boldsymbol{\sigma}_T \cdot (-\mathbf{u}_T) = \big(2M_{\alpha\beta} u_{T\alpha} u_{T\beta} H(\sigma_N)\big)^{1/2}. \tag{3.24}$$

4. Limit analysis theorems in the presence of friction when normal stresses are prescribed on Γ_2

In this section of the paper we shall formulate and prove both the lower and upper bound theorems in the presence of friction on Γ_2. Here the particular case of friction is examined, namely normal stresses are prescribed.

Let us set

$$\mathscr{K}_s = \{\boldsymbol{\tau} = (\tau_{ij}) | \tau_{ij,j} + b_i = 0, \quad \boldsymbol{\tau} \in C, \Omega; \ \tau_{ij} n_j = \mu^s p_i^0, \ \Gamma_1; \ \tau_N = F_N \quad h(\boldsymbol{\tau}_T) \leqslant 0, \ \Gamma_2\}, \tag{4.1}$$

where $h(\boldsymbol{\tau}_T) = g(F_N, \boldsymbol{\tau}_T)$, and F_N is a given function defined over Γ_2.

A stress field $\boldsymbol{\tau} \in \mathscr{K}_s$ is called *statically admissible*.

We set

$$\mathscr{K}_v = \{\mathbf{v} = (v_i) | \mathbf{v} = 0, \Gamma_0\} \tag{4.2}$$

or, in the case of incompressible materials,

$$\mathscr{K}_v = \{\mathbf{v} = (v_i) | v_{i,i} = 0, \Omega; \ \mathbf{v} = 0, \Gamma_0\}. \tag{4.3}$$

A velocity field $\mathbf{v} \in \mathscr{K}_v$ is called *kinematically admissible*.

A *kinematical load multiplier* μ^k is now defined as follows:

$$\mu^k(\mathbf{v}) = \frac{D(\boldsymbol{\epsilon}(\mathbf{v})) + J(\mathbf{v}) - \int\limits_{\Omega} \mathbf{b} \cdot \mathbf{v} dx - \int\limits_{\Gamma_2} F_N v_N d\Gamma}{\int\limits_{\Gamma_2} \mathbf{p}^0 \cdot \mathbf{v} d\Gamma}, \tag{4.4}$$

where the functional

$$J(\mathbf{v}) = \int\limits_{\Gamma_2} j(F_N, \mathbf{v}_T) d\Gamma \tag{4.5}$$

is convex and subdifferentiable. This functional represents the total friction dissipation

A triple $(\mu, \boldsymbol{\sigma}, \mathbf{u})$ is called a complete solution if $\boldsymbol{\sigma} \in \mathscr{K}_s$, $\mathbf{u} \in \mathscr{K}_v$, μ is the associated load multiplier, while $\boldsymbol{\sigma}$ and $\boldsymbol{\epsilon}(\mathbf{u})$ are interrelated by the flow rule $(2.7)_1$ or $(2.7)_2$. Moreover, on Γ_2, $\boldsymbol{\sigma}_T$ and $\mathbf{u}_T$ are interrelated by the friction law, that is $(-\mathbf{u}_T) \in \partial h(\boldsymbol{\tau}_T)$.

4.1. Lower bound theorem

We formulate this theorem as the maximization problem

$$\text{(4.6)} \qquad \left| \begin{array}{l} \text{find} \\ \quad \sup\limits_{\tau \in \mathscr{K}_s} \mu^s(\boldsymbol{\tau}) \end{array} \right.$$

where the closed convex set $\mathscr{K}_s$ is given by the relations (4.1).

Now we shall prove that $\mu^s \leqslant \mu$. For this purpose we use Eq. (2.24) where $\boldsymbol{\tau} \in \mathscr{K}_s$, which now yields

$$\int\limits_{\Omega} (\boldsymbol{\tau}-\boldsymbol{\sigma}) \cdot \boldsymbol{\epsilon}(\mathbf{u}) dx = (\mu^s - \mu) \int\limits_{\Gamma_1} \mathbf{p}^0 \cdot \mathbf{u} d\Gamma + \int\limits_{\Gamma_2} [(\tau_N - \sigma_N) u_N + (\boldsymbol{\tau}_T - \boldsymbol{\sigma}_T) \cdot \mathbf{u}_T] d\Gamma, \qquad \forall \boldsymbol{\tau} \in \mathscr{K}_s.$$

Taking account of the inequality (3.11), which particularly holds for each $\boldsymbol{\tau} \in \mathscr{K}_s$, and knowing that $\tau_N = \sigma_N = F_N$ on Γ_2, we arrive at the inequality $\mu^s - \mu \leqslant 0$.

4.2. Upper bound theorem

This theorem is formulated in the form of the minimization problem

$$\text{(4.7)} \qquad \left| \begin{array}{l} \text{find} \\ \quad \inf\limits_{\mathbf{v} \in \mathscr{K}_v} \mu^k(\mathbf{v}). \end{array} \right.$$

It is sufficient to demonstrate that $\mu \leqslant \mu^k$. The inequality (2.30), now valid for each $\mathbf{v} \in \mathscr{K}_v$, after integration by parts yields

$$\text{(4.8)} \qquad D(\boldsymbol{\epsilon}(\mathbf{v})) - D(\boldsymbol{\epsilon}(\mathbf{u})) + \int\limits_{\Gamma_2} (-\boldsymbol{\sigma}_T) \cdot (\mathbf{v}_T - \mathbf{u}_T) d\Gamma \geqslant \int\limits_{\Omega} \mathbf{b} \cdot (\mathbf{v}-\mathbf{u}) dx + \mu \int\limits_{\Gamma_1} \mathbf{p}^0 \cdot (\mathbf{v}-\mathbf{u}) d\Gamma + \int\limits_{\Gamma_2} F_N (v_N - u_N) d\Gamma, \qquad \forall \mathbf{v} \in \mathscr{K}_v.$$

The subdifferentiability of the functional J implies

$$\text{(4.9)} \qquad J(\mathbf{v}) - J(\mathbf{u}) \geqslant \int\limits_{\Gamma_2} (-\boldsymbol{\sigma}_T) \cdot (\mathbf{v} - \mathbf{u}_T) d\Gamma.$$

The inequality (4.9) certainly holds for each $\mathbf{v} \in \mathscr{K}_v$. Substituting the relation (4.9) into the inequality (4.8) and taking account of the inequality (2.17) and of the definition (4.4), we obtain the desired results, that is $\mu^k \geqslant \mu$.

REMARK 4.1. In the numerator of the expression (4.4) the term representing the power of normal stresses F_N is present. The definition of the kinematical load multiplier can be modified by dropping this term, that is

$$\text{(4.10)} \qquad \tilde{\mu}^k(\mathbf{v}) = \frac{D(\boldsymbol{\epsilon}(\mathbf{v})) + J(\mathbf{v}) - \int\limits_{\Omega} \mathbf{b} \cdot \mathbf{v} dx}{\int\limits_{\Gamma_1} \mathbf{p}^0 \cdot \mathbf{v} d\Gamma}.$$

We observe that in the absence of body forces the numerator of (4.10) represents the total dissipation caused by plastic flow and friction. It can readily be demonstrated that the lower bound theorem remains valid without any additional assumptions. However, for the upper bound theorem to hold we must assume that $\mathbf{v} \in \mathscr{K}_v(\mathbf{u})$, where

$$\mathscr{K}_v(\mathbf{u}) = \{\mathbf{v} \in \mathscr{K}_v | v_N = u_N, \quad \text{on} \quad \Gamma_2\}. \tag{4.11}$$

Thus we see that in this case the set of admissible velocity fields depends on an unknown limit state velocity $\mathbf{u}$. Therefore, from the point of view of applications the definition (4.10) of the kinematical load multiplier is of lower value than the formula (4.4). Rigorously, the kinematic approach in the case of the definition (4.4) would have to be combined with a fixed point theorem or an iterative procedure, see the next section.

5. Limit analysis in the presence of Signorini's boundary condition with friction

Let as proceed to the formulation of the theory of limit analysis in the case when on Γ_2 Signorini's conditions $(2.3)_2$ in combination with a general friction law (3.9), or equivalently (3.13), have to be satisfied. The difficulty in formulating limit analysis theorems issues from the non-associated character of the friction law. Previous attempts to generalize the lower and upper bound theorems for nonstandard plastic materials are described in the papers [20–23, 29]. Yet, in these papers only some estimates of a load multiplier are presented. These estimates are valid for a restricted class of nonstandard materials.

If the lower and upper bound theorems are applied in the usual manner, then the example given by SALENÇON [24] clearly exhibits that even in the presence of Coulomb's friction we can obtain $\mu^s > \mu^k$. The theory proposed in this section excludes inherently such conclusion.

Let us set

$$K_s(\theta_N) = \{\boldsymbol{\tau} = (\tau_{ij}) | \tau_{ij,j} + b_i = 0, \ \boldsymbol{\tau} \in C, \Omega; \ \tau_{ij} n_j = \mu^s p_i^0, \ \Gamma_1; \quad \tau_N \leqslant 0, \ \boldsymbol{\tau}_T \in K(\theta_N), \Gamma_2\}, \tag{5.1}$$

where the convex set $K(\theta_N)$ is defined by the formula (3.9).

The static approach to the problem of limit analysis under consideration is now proposed. Let θ_N be a sufficiently regular function defined over Γ_2, such that $\theta_N \leqslant 0$ and $g(\theta_N, \tau_T) \leqslant 0$ makes sense. Thus we can consider the convex maximization problem

$$\left| \begin{array}{l} \text{find} \\ \qquad \sup\limits_{\boldsymbol{\tau} \in K_s(\theta_N)} \mu^s(\boldsymbol{\tau}) \end{array} \right. \tag{5.2}$$

since the set $K_s(\theta_N)$ is convex.

A solution $\boldsymbol{\sigma}$ of the problem (5.2) depends on θ_N, that is $\boldsymbol{\sigma} = \boldsymbol{\sigma}(\theta_N)$. Obviously, the load multiplier μ^s associated with (5.2) depends on $\boldsymbol{\sigma}$ and θ_N, that is $\mu^s = \mu^s(\theta_N, \boldsymbol{\sigma})$.

For a tensor field $\boldsymbol{\tau} = (\tau_{ij})$ belonging to an appropriate space we define $T_N(\boldsymbol{\tau})$ in the following manner:

$$\int\limits_{\Gamma_2} T_N(\boldsymbol{\tau}) w_N d\Gamma = \int\limits_{\Omega} (\tau_{ij} \varepsilon_{ij}(\mathbf{w}) - b_i w_i) d\mathbf{x}, \quad \forall \mathbf{w} = (w_i) \quad \text{such that} \quad \mathbf{w}|_{\Gamma \setminus \Gamma_2} = 0 \quad \text{and} \quad \mathbf{w}_{T|\Gamma_2} = 0. \tag{5.3}$$

If $\boldsymbol{\tau}$ satisfies the equilibrium equations (2.11), then $T_N(\boldsymbol{\tau}) = \tau_N$, on Γ_2. In this manner a nonlinear operator N: $\theta_N \to T_N(\boldsymbol{\tau})$ is implicitly defined.

Let $\boldsymbol{\sigma}$ be a solution of the problem (5.2) for some θ_N. Suppose that $\theta_N = T_N(\boldsymbol{\sigma})$. Then the stress field $\boldsymbol{\sigma}$ is called *a static solution of the limit analysis in the case of Signorini's boundary conditions with friction.*

Such a definition of the static solution is correct since then $\theta_N = T_N(\boldsymbol{\sigma}) = N(\theta_N)$. Hence we infer that $\theta_N = \sigma_N$ is a fixed point of the operator N. We observe that if $\sigma_N = N(\sigma_N)$, then we have the following *counterpart of the lower bound theorem*

$$\mu^s(\boldsymbol{\tau}) \leqslant \mu^s(\sigma_N, \boldsymbol{\sigma}) \qquad \forall \boldsymbol{\tau} \in K_s(\sigma_N), \tag{5.4}$$

where $\mu^s(\boldsymbol{\tau})$ enters the definition of the convex set $K_s(\sigma_N)$, see the formula (5.1). A field $\boldsymbol{\tau} \in K_s(\sigma_N)$ is *statically admissible.*

If $\tilde{\mu}(\tilde{\boldsymbol{\sigma}})$ is a solution of the maximization problem

$$\left|\begin{array}{l}\text{find} \\ \qquad \sup\limits_{\boldsymbol{\tau}\in K_s} \mu^s(\boldsymbol{\tau}),\end{array}\right. \tag{5.5}$$

where

$$K_s = \{\boldsymbol{\tau} = (\tau_{ij}) | \tau_{ij,j} + b_i = 0, \quad \boldsymbol{\tau} \in C,\ \Omega;\ \tau_{ij}n_j = \mu^s p_i^0,\ \Gamma_1; \\ \tau_N \leqslant 0,\ g(\tau_N, \boldsymbol{\tau}_T) \leqslant 0,\ \Gamma_2\} \tag{5.6}$$

then obviously we have $\mu_s(\sigma_N, \boldsymbol{\sigma}) \leqslant \tilde{\mu}(\tilde{\boldsymbol{\sigma}})$.

The problem (5.5) would just represent the lower bound theorem formulated in the "usual" manner provided that K_s is closed and convex, see Sects. 2 and 4 of this paper. The example given in Ref. [24] indicates that such an approach is fallacious.

If σ_N is as above, that is $\sigma_N = N(\sigma_N)$, then we define a *kinematically admissible* velocity field as an arbitrary field $\mathbf{v} \in K_v$, where K_v is given by the relations (2.28) or (2.29). The kinematical load multiplier is now defined by

$$\mu^k(\sigma_N, \mathbf{v}) = \frac{D\big(\boldsymbol{\epsilon}(\mathbf{v})\big) + J(\sigma_N, \mathbf{v}) - \int\limits_{\Omega} \mathbf{b}\cdot\mathbf{v}\,d\mathbf{x} - \int\limits_{\Gamma_2} \sigma_N v_N\,d\Gamma}{\int\limits_{\Gamma_1} \mathbf{p}^0\cdot\mathbf{v}\,d\Gamma}, \tag{5.7}$$

where

$$J(\sigma_N, \mathbf{v}) = \int\limits_{\Gamma_2} j(\sigma_N, \mathbf{v})\,d\Gamma. \tag{5.8}$$

A counterpart of the upper bound theorem is formulated as a minimization problem:

$$\left|\begin{array}{l}\text{find} \\ \qquad \inf\limits_{\mathbf{v}\in K_v} \mu^k(\sigma_N, \mathbf{v}).\end{array}\right. \tag{5.9}$$

Our previous considerations imply that $\mu^s(\sigma_N, \boldsymbol{\sigma}) \leqslant \mu^k(\sigma_N, \mathbf{v}), \forall \mathbf{v} \in K_v$. In general, neither μ nor $\boldsymbol{\sigma}$ are unique. As we know, a velocity field $\mathbf{u}$ solving the problem (5.9) is not unique even in frictionless cases. In the case of Coulomb's friction the limit load multiplier μ is unique for a sufficiently small coefficient of friction.

It seems reasonable to define a limit load multiplier μ, in the general case of friction examined in the present section, as follows:

$$\mu = \mu^s(\sigma_N, \boldsymbol{\sigma}) = \mu^k(\sigma_N, \mathbf{u}). \tag{5.10}$$

where $\mathbf{u}$ is a solution of the problem (5.9).

REMARK 5.1. As we have seen the kinematic approach proposed here for Signorini's problem with friction is coupled with the static approach. I think that this conclusion is not surprising since the density of dissipation caused by friction depends not only on the tangential components of a velocity vector but also explicitly on the normal component of the stress vector.

REMARK 5.2. The nonlinear operator N defined previously is not known explicitly therefore, from the viewpoint of applications, the method of the construction of a static field $\boldsymbol{\sigma}$ is not effective. However, an iterative procedure overcomes this drawback. For this purpose let us take a function $\tau_N^{(0)}$ defined over Γ_2 and such that $\tau_N^{(0)} \leqslant 0$ and $K(\tau_N^{(0)})$ makes sense. In the next step we solve the convex problem

$$\left| \begin{array}{l} \text{find} \\ \quad \sup\limits_{\boldsymbol{\tau} \in K_s(\tau_N^{(0)})} \mu_1^s(\boldsymbol{\tau}). \end{array} \right. \tag{5.11}$$

A solution $\boldsymbol{\tau}^1$ of the problem (5.11) yields $\tau_N^{(1)}$ on Γ, and hence also on Γ_2. In this manner we obtain a sequence of maximization problems

$$\left| \begin{array}{l} \text{find} \\ \quad \sup\limits_{\boldsymbol{\tau} \in K_s(\tau_N^{(n-1)})} \mu_n^s(\boldsymbol{\tau}), \qquad n = 1, 2, \ldots, \end{array} \right. \tag{5.12}$$

where

$$K_s(\tau_N^{(n-1)}) = \{\boldsymbol{\tau} | \tau_{ij,j} + b_i = 0, \ \boldsymbol{\tau} \in C, \Omega; \ \tau_{ij}n_j = \mu_n p_i^0, \ \Gamma_1; \ \tau_N \leqslant 0, \ \boldsymbol{\tau}_T \in K(\tau_N^{(n-1)}), \ \Gamma_2\} \tag{5.13}$$

and

$$K(\tau_N^{(n-1)}) = \{\boldsymbol{\tau}_T | g(\tau_N^{(n-1)}, \boldsymbol{\tau}_T) \leqslant 0\}. \tag{5.14}$$

Hence we have a sequence $\boldsymbol{\tau}^n$, $n = 1, 2, \ldots$. In the limit, if it exists, we obtain $\lim\limits_{n\to\infty} \boldsymbol{\tau}^n = \boldsymbol{\sigma}$.

The ideas presented in this section require further and more rigorous, in the mathematical sense, studies.

REMARK 5.3. If Γ_2 is an interface with friction, then on this surface we have, see Remark 2.4,

$$[\![\mathbf{u}_T]\!] \in \partial\chi_{K(\sigma_N)}(\boldsymbol{\sigma}_T), \qquad \text{on } \Gamma_2 \tag{5.15}$$

while Signorini's conditions read

$$[\![u_N]\!] \leqslant 0, \qquad \sigma_N \leqslant 0, \qquad \sigma_N[\![u_N]\!] = 0, \tag{5.16}$$

where $\boldsymbol{\sigma}_T = \boldsymbol{\sigma}_T^{(1)} = -\boldsymbol{\sigma}_T^{(2)}$.

REMARK 5.4. A similar procedure can be applied to a formulation of limit analysis theorems in the case of incompressible materials obeying the non-associated flow rule. However, this problem is not investigated here.

The ideas developed by LABORDE [16] for elastic-plastic materials will be studied separately.

REMARK 5.5. Nonlocal Coulomb's friction has been introduced by DUVAUT [7], see also [18]. Limit analysis theorems in the presence of nonlocal friction will be studied in the future. However we observe that nonlocal friction laws can be formulated in a general manner, similarly to constitutive relations of nonlocal plasticity [8, 9].

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