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A SHEEPSHANKS' PROBLEM (1866).

[From the *Messenger of Mathematics*, vol. IV. (1875), pp. 34—36.]

Apply the formulæ of elliptic motion to determine the motion of a body let fall from the top of a tower at the equator.

The earth is regarded as rotating with the angular velocity ω round a fixed axis, so that the body is in fact projected from the apocentre with an angular velocity $=\omega$; and we write α for the equatorial radius, β for the height of the tower; then g denoting the force of gravity, and μ, h, n, a, e, θ, as in the theory of elliptic motion, we have

$$\mu = \quad n^2a^3 \quad = g\alpha^2,$$

$$h = (\alpha+\beta)^2\,\omega = na^2\sqrt{(1-e^2)},$$

$$\alpha+\beta \quad = a\,(1+e);$$

whence

$$(\alpha+\beta)^4\,\omega^2 = g\alpha^2 a\,(1-e^2),$$

$$(\alpha+\beta) \quad = \quad a\,(1+e),$$

$$\frac{(\alpha+\beta)^3\,\omega^2}{g\alpha^2} = \frac{\alpha\omega^2}{g}\left(1+\frac{\beta}{\alpha}\right)^3 = 1-e,$$

where $\dfrac{\alpha\omega^2}{g}=$ ratio of centrifugal force to gravity,

$$= \tfrac{1}{289},$$

so that $1-e$ is small;

$$r = \frac{a\,(1-e^2)}{1-e\cos\theta}, \quad = \frac{(\alpha+\beta)\,(1-e)}{1-e\cos\theta},$$

whence

$$1-e\cos\theta \quad = \frac{(\alpha+\beta)\,(1-e)}{r}.$$

Suppose

$$r=\alpha,$$

$$1-e\cos\theta=(1-e)\left(1+\frac{\beta}{\alpha}\right)=1-e+\frac{\beta}{\alpha}(1-e),$$

which is nearly

$$=1-e,$$

that is, θ is small, and therefore approximately

$$1-e+\tfrac{1}{2}e\theta^2=1-e+\frac{\beta}{\alpha}(1-e),$$

or

$$\theta^2=\frac{2\beta}{\alpha}\frac{1-e}{e};$$

we then have

$$r^2d\theta=hdt,\ \text{ or }\ dt=\frac{r^2d\theta}{h}=\frac{(\alpha+\beta)^2(1-e)^2}{(\alpha+\beta)^2\,\omega\,(1-e\cos\theta)^2}d\theta$$

$$=\frac{(1-e)^2}{\omega^2(1-e\cos\theta)^2}d\theta$$

$$=\frac{(1-e)^2}{\omega\,(1-e+\frac{1}{2}e\theta^2)^2}d\theta$$

$$=\frac{1}{\omega\left(1+\frac{\frac{1}{2}e}{1-e}\theta^2\right)^2}d\theta$$

$$=\frac{1}{\omega}\left(1-\frac{e}{1-e}\theta^2\right)d\theta;$$

that is,

$$\omega\,dt=\left(1-\frac{e}{1-e}\theta^2\right)d\theta.$$

Integrating, we have

$$\theta\left(1-\frac{\frac{1}{3}e\theta^2}{1-e}\right)=\omega t,$$

where $\omega t=$ earth's rotation in time t, $=\phi$ suppose; therefore

$$\theta\left(1-\frac{\frac{1}{3}e\theta^2}{1-e}\right)=\phi,$$

hence, if θ be as above, the angle described in falling to the surface,

$$\frac{e\theta^2}{1-e}=\frac{2\beta}{\alpha},$$

$$\theta\left(1-\tfrac{2}{3}\frac{\beta}{\alpha}\right)=\phi,$$

$$\theta-\phi=\tfrac{2}{3}\frac{\beta}{\alpha}\theta=\tfrac{2}{3}\frac{\beta}{\alpha}\sqrt{\left(\frac{2\beta}{\alpha}\frac{1-e}{e}\right)}.$$

Writing herein

$$\frac{1-e}{e},\ =1-e,\ =\frac{\alpha\omega^2}{g},$$

this is

$$=\tfrac{2}{3}\frac{\beta}{\alpha}\sqrt{\left(\frac{2\beta}{\alpha}\ \frac{\alpha\omega^2}{g}\right)}=\frac{(2\beta)^{\frac{3}{2}}}{3\alpha^{\frac{3}{2}}}\sqrt{\left(\frac{\alpha\omega^2}{g}\right)},$$

viz.

$$\theta-\phi=\frac{2^{\frac{3}{2}}.\,\beta^{\frac{3}{2}}}{3\alpha^{\frac{3}{2}}}\sqrt{\left(\frac{\alpha\omega^2}{g}\right)},$$

whence

$$\alpha\,(\theta-\phi)=\frac{2\,\sqrt{(2)}}{3}\sqrt{\left(\frac{\beta}{\alpha}\right)}\sqrt{\left(\frac{\alpha\omega^2}{g}\right)}\,\beta,$$

or say

$$\frac{\alpha\,(\theta-\phi)}{\beta}=\frac{2\,\sqrt{(2)}}{3}\sqrt{\left(\frac{\beta}{\alpha}\right)}\sqrt{\left(\frac{\alpha\omega^2}{g}\right)},$$

where $\alpha\,(\theta-\phi)$ is the distance at which the body falls from the foot of the tower. Substituting for $\sqrt{\left(\frac{\alpha\omega^2}{g}\right)}$ its value, $=\frac{1}{17}$, we have

$$\frac{\alpha\,(\theta-\phi)}{\beta}=\frac{2\,\sqrt{(2)}}{51}\sqrt{\left(\frac{\beta}{\alpha}\right)},\ =\cdot 056\sqrt{\left(\frac{\beta}{\alpha}\right)}.$$