599.

A SMITH'S PRIZE DISSERTATION.

[From the *Messenger of Mathematics*, vol. IV. (1875), pp. 157—160.]

Write a dissertation on Bernoulli's Numbers and their use in Analysis.

The function $\dfrac{t}{e^t-1}+\tfrac{1}{2}t$ is an *even* function of t, as appears by expressing it in the form

$$\tfrac{1}{2}t\,\frac{e^t+1}{e^t-1},\quad =\tfrac{1}{2}t\,\frac{e^{\frac{1}{2}t}+e^{-\frac{1}{2}t}}{e^{\frac{1}{2}t}-e^{-\frac{1}{2}t}},$$

and its value for $t=0$ being obviously $=1$, we may write

$$\frac{t}{e^t-1}+\tfrac{1}{2}t=1+B_1\frac{t^2}{1\,.\,2}-B_2\frac{t^4}{1\,.\,2\,.\,3\,.\,4}+\text{\&c.};$$

or, what is the same thing,

$$\frac{t}{e^t-1}=1-\tfrac{1}{2}t+B_1\frac{t^2}{1\,.\,2}-B_2\frac{t^4}{1\,.\,2\,.\,3\,.\,4}+\ldots+(-)^{n-1}B_n\frac{t^{2n}}{1\,.\,2\ldots 2n}+\ldots,$$

where the several coefficients B_1, B_2, B_3, &c., are, as is at once seen, rational fractions, and, as it may be shown, are all of them positive. These numerical coefficients B_1, B_2, B_3, &c., are called Bernoulli's numbers.

There is no difficulty in calculating directly the first few terms; viz. we have

$$\begin{aligned}\frac{t}{e^t-1}=\frac{1}{1+(\frac{1}{2}t+\frac{1}{6}t^2+\frac{1}{24}t^3+\ldots)}&=1-t\,(\tfrac{1}{2}+\tfrac{1}{6}t+\tfrac{1}{24}t^2+\tfrac{1}{120}t^3+\ldots)\\&\quad+t^2(\tfrac{1}{4}+\tfrac{1}{6}t+\tfrac{5}{72}t^2+\ldots)\\&\quad-t^3(\tfrac{1}{8}+\tfrac{1}{8}t+\ldots)\\&\quad+t^4(\tfrac{1}{16}+\ldots)\\&=1+t\,(-\tfrac{1}{2})+t^2(-\tfrac{1}{6}+\tfrac{1}{4},\ =+\tfrac{1}{12})+t^3(-\tfrac{1}{24}+\tfrac{1}{6}-\tfrac{1}{8}=0)\\&\qquad+t^4(-\tfrac{1}{120}+\tfrac{5}{72}-\tfrac{1}{8}+\tfrac{1}{16}=-\tfrac{1}{720})+\ldots,\end{aligned}$$

viz.

$$=1-\tfrac{1}{2}t+\tfrac{1}{12}t^2-\tfrac{1}{720}t^4+\ldots,$$

which is therefore

$$=1-\tfrac{1}{2}t+\tfrac{1}{2}B_1t^2-\tfrac{1}{24}B_2t^4+\dots,$$

and consequently

$$B_1=\tfrac{1}{6},\quad B_2=\tfrac{1}{30},$$

and so a few more terms might have been found.

But a more convenient method is to express the numbers in terms of the differences of 0^m by means of a general formula for the expansion of a function of e^t, viz. this is

$$\phi(e^t)=\phi(1+\Delta)\,e^{t.0},$$

where

$$e^{t.0}=0^0+\frac{t}{1}0^1+\frac{t^2}{1\,.\,2}0^2+\frac{t^3}{1\,.\,2\,.\,3}0^3+\&c.,$$

and the $\phi(1+\Delta)$ is to be applied to the terms 0^0, 0^1, 0^2, 0^3, &c. We have thus

$$\frac{t}{e^t-1}=\frac{\log(e^t)}{e^t-1}$$

$$=\frac{\log(1+\Delta)}{\Delta}e^{t.0}$$

$$=\frac{\log(1+\Delta)}{\Delta}\left\{0^0+\frac{t}{1}0^1+\frac{t^2}{1\,.\,2}0^2+\&c.\dots+\frac{t^{2n-1}}{1\,.\,2\dots 2n-1}0^{2n-1}+\frac{t^{2n}}{1\,.\,2\dots 2n}0^{2n}+\dots\right\}.$$

We have, as may be at once verified,

$$\frac{\log(1+\Delta)}{\Delta}0^0=1,\quad \frac{\log(1+\Delta)}{\Delta}0^1=-\tfrac{1}{2},$$

and by what precedes, since the coefficient of every higher odd power of t vanishes,

$$\frac{\log(1+\Delta)}{\Delta}0^{2n-1}=0\,;$$

and then, by comparing the even powers of t,

$$(-)^{n-1}B_n=\frac{\log(1+\Delta)}{\Delta}0^{2n},$$

that is,

$$(-)^{n-1}B_n=\left(1-\tfrac{1}{2}\Delta+\tfrac{1}{3}\Delta^2\dots+\frac{1}{2n+1}\Delta^{2n}\right)0^{2n},$$

the series for $\dfrac{\log(1-\Delta)}{\Delta}$ being stopped at this point since $\Delta^{2n+1}0^{2n}=0$, &c. For instance, in the case $n=1$, we have

$$\begin{aligned}B_1=(1-\tfrac{1}{2}\Delta+\tfrac{1}{3}\Delta^2)\,0^2= \;& 0^2\\ & -\tfrac{1}{2}(1^2-0^2)\\ & +\tfrac{1}{3}(2^2-2\,.\,1^2+0^2)\\ =\;& -\tfrac{1}{2}+\tfrac{2}{3},\ =\tfrac{1}{6}\text{ as above.}\end{aligned}$$

The formula shows, not only that B_n is a rational fraction but that its denominator is at most = least common multiple of the numbers $2, 3, \ldots, 2n+1$; the actual denominator of the fraction in its least terms is, however, much less than this, there being as to its value a theorem known as Staudt's theorem. It does not obviously show that the Numbers are positive, or afford any indication of the rate of increase of the successive terms of the series.

These last requirements are satisfied by an expression for B_n as the sum of an infinite numerical series, which expression is obtained by means of the function $\cot\theta$, as follows:

We have

$$\frac{t}{e^t-1}+\tfrac{1}{2}t, \quad = \tfrac{1}{2}t\,\frac{e^{\frac{1}{2}t}+e^{-\frac{1}{2}t}}{e^{\frac{1}{2}t}-e^{-\frac{1}{2}t}} = 1 + B_1\frac{t^2}{1\,.\,2} - B_2\frac{t^4}{1\,.\,2\,.\,3\,.\,4} + \text{\&c.},$$

or, writing herein $t = 2i\theta$ $\{i = \sqrt{(-1)}$ as usual$\}$, this is

$$\theta\cot\theta = 1 - B_1\frac{2^2\theta^2}{1\,.\,2} - B_2\frac{2^4\theta^4}{1\,.\,2\,.\,3\,.\,4} - \text{\&c.}$$

But we have

$$\log\sin\theta = \log\theta + \log\left(1-\frac{\theta^2}{\pi^2}\right) + \log\left(1-\frac{\theta^2}{2^2\pi^2}\right) + \ldots,$$

and thence, by differentiation,

$$\begin{aligned}\theta\cot\theta &= 1 - \frac{2\theta^2}{\pi^2}\left\{\frac{1}{1^2\left(1-\frac{\theta^2}{1^2\pi^2}\right)} + \frac{1}{2^2\left(1-\frac{\theta^2}{2^2\pi^2}\right)} + \ldots\right\}\\ &= 1 - \frac{2\theta^2}{\pi^2}\left\{\frac{1}{1^2}+\frac{1}{2^2}+\text{\&c.}\right\}\\ &\quad - \frac{2\theta^4}{\pi^4}\left\{\frac{1}{1^4}+\frac{1}{2^4}+\text{\&c.}\right\}\\ &\quad - \text{\&c.}\end{aligned}$$

Hence

$$B_n\frac{2^{2n}}{1\,.\,2\ldots 2n} = \frac{2}{\pi^{2n}}\left\{\frac{1}{1^{2n}}+\frac{1}{2^{2n}}+\ldots\right\},$$

that is,

$$B_n = \frac{2\,(1\,.\,2\,.\,.\,2n)}{(2\pi)^{2n}}\left(\frac{1}{1^{2n}}+\frac{1}{2^{2n}}+\ldots\right),$$

showing first, that B_n is positive, and next, that it rapidly increases with n, viz. n being large, we have

$$B_n = \frac{2\,(1\,.\,2\ldots 2n)}{(2\pi)^{2n}},$$

or, instead of $1\,.\,2\ldots 2n$ writing its approximate value $\sqrt{(2\pi)}\,.\,(2n)^{2n+\frac{1}{2}}e^{-2n}$, this is

$$B_n = 4\sqrt{(n\pi)}\left(\frac{n}{\pi e}\right)^{2n}.$$

The result may of course be considered from the opposite point of view, as giving a determination of the sum $\frac{1}{1^{2n}}+\frac{1}{2^{2n}}+\&c.\ldots$ in terms of Bernoulli's Numbers, assumed to be known, viz. we thus have

$$\frac{1}{1^{2n}}+\frac{1}{2^{2n}}+\ldots=\frac{(2\pi)^{2n}}{2\,(1\,.\,2\ldots 2n)}\,B_n.$$

For instance, $n=1$,

$$\frac{1}{1^2}+\frac{1}{2^2}+\ldots=\frac{(2\pi)^2}{2\,.\,1\,.\,2}\cdot\tfrac{1}{6},\ =\frac{\pi^2}{6},$$

and this is one and a good instance of the use of Bernoulli's Numbers in Analysis.

Another and very important one is in the summation of a series, or say in the determination of $\Sigma u_x,\ =u_0+u_1+\ldots+u_{x-1}$; viz. starting from

$$\frac{1}{e^t-1}=\frac{1}{t}-\tfrac{1}{2}+\frac{B_1}{1\,.\,2}\,t-\frac{B_2}{1\,.\,2\,.\,3\,.\,4}\,t^3+\&c.,$$

and writing herein $t=d_x$, and therefore

$$\frac{1}{e^t-1}=\frac{1}{e^{d_x}-1},\ =\frac{1}{\Delta}\text{ or }\Sigma,$$

and applying each side to a function u_x of x, we have

$$\Sigma u_x=C+\int dx\,u_x-\tfrac{1}{2}u_x+\frac{B_1}{1\,.\,2}\,d_xu_x-\frac{B_2}{1\,.\,2\,.\,3\,.\,4}\,d_x^{\,3}u_x+\ldots,$$

or taking the two sides each between the integer limits a, x,

$$u_a+u_{a+1}\ldots+u_{x-1}=\int_a^x dxu_x-\tfrac{1}{2}(u_x-u_a)+\frac{B_1}{1\,.\,2}\left(d_xu_x\right)_a^x-\frac{B_2}{1\,.\,2\,.\,3\,.\,4}\left(d_x^{\,3}u_x\right)_a^x+\ldots,$$

where if u_x is a rational and integral function the series on the right-hand side is finite. If for instance $u_x=x$, the equation is

$$a+(a+1)\ldots+(x-1)=\tfrac{1}{2}(x^2-a^2)-\tfrac{1}{2}(x-a),$$

viz.

$$\{1+2\ldots+(x-1)\}-\{1+2\ldots+(a-1)\}=\tfrac{1}{2}(x^2-n)-\tfrac{1}{2}(a^2-a),$$

which is right.

Applying the formula to the function $\log x$, we deduce theorems as to the Γ-function; and it is also interesting to apply it to $\frac{1}{x}$.

The above is given as a specimen of what might be expected in an examination: I remark as faults the omission to make it clear that B_n is a rational fraction; and the giving the series-formula as a formula for the convenient calculation of B_n. The omission to give the first-mentioned straightforward process of development strikes me as curious.