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Remarks on the semigroups method in linear thermoviscoelasticity

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THIS PAPER presents an example in which the semigroups theory is applied to the problem of existence and uniqueness for the equations arising in linear thermoelasticity with fading memory. The boundary-initial history value problem will be transformed to an initial value problem $dx/dt = Ax+f$, $x(0) = x_0$ in the Hilbert space X, where the operator A will satisfy the assumptions of the Hille–Yosida theorem.

W pracy przedstawiono przykład, w którym teorię półgrup zastosowano do zagadnienia istnienia i jednoznaczności rozwiązań równań liniowej termosprężystości z zanikającą pamięcią. Zagadnienie brzegowo-początkowe z historią zostało przekształcone na zagadnienie początkowe $dx/dt = Ax+f$, $x(0) = x_0$ w przestrzeni Hilberta X, przy czym operator A spełnia założenia twierdzenia Hille–Yosidy.

В работе представлен пример, в котором теория полугрупп применена к задачам существования и единственности решений линейной термоупругости с исчезающей памятью. Краево-начальная задача с историей преобразована к начальной задаче $dx/dt = Ax+f$, $x(0) = x_0$ в пространстве Гильберта X, причем оператор A удовлетворяет предположениям теоремы Хилле–Иосида.

WE ASSUME that the body composed of an inhomogeneous, anisotropic linear thermoviscoelastic material occupies a bounded region $B \subset R^3$ with a smooth boundary ∂B and that the reference configuration is a natural state in which the stress is zero and the base temperature $\theta_0 > 0$. Moreover, we assume that the Cauchy stress tensor $\bar{t}$ and specific entropy e are given by the functional H

$$\big(\bar{t}(x, t), e(x, t)\big) = H\big(\nabla u(x, \cdot)^t, \theta(x, \cdot)^t\big),$$

where the function $\nabla u(x, s)^t = \nabla u(x, t-s)\ \forall\, s \geqslant 0$ is a history of the displacement gradient and the function $\theta(x, s)^t = \theta(x, t-s)\ \forall\, s \geqslant 0$ is a history of the temperature difference from θ_0. From the general fading memory theory the constitutive relations are given in the following form:

$$\bar{t}(x, t) = g(x, 0)\nabla u(x, t) - l(x, 0)\theta(x, t) + \int_0^\infty [g'(x, s)\nabla u(x, t-s) - l'(x, s)\theta(x, t-s)]\,ds,$$

$$r(x)e(x, t) = l(x, 0)\nabla u(x, t) + r(x)c(x, 0)\theta(x, t)/\theta_0 + \int_0^\infty [l'(x, s)\nabla u(x, t-s) + r(x)c'(x, s)\theta(x, t-s)/\theta_0]\,ds,$$

where the functions $g(x, s)$, $l(x, s)$, $c(x, s)$ for $s \geqslant 0$ are the relaxation tensors of fourth, second and zero order, respectively; $r(x)$ is the mass density ("$'$" denotes the derivative $\partial/\partial s$).

Let us assume we have also the equation for the heat flux vector

$$q(x, t) = -k(x)\nabla\theta(x, t),$$

where $k(x)$ is the thermal conductivity.

From the linear thermoelasticity equations

$$\operatorname{div}\bar{t}(x, t) = r(x)[\ddot{u}(x, t)-f(x, t], \quad f(x, t)\text{—body forces},$$
$$\theta_0 r(x)\dot{e}(x, t)+\operatorname{div}q(x, t) = 0$$

and the constitutive relations we obtain the following linear thermoviscoelasticity equations:

$$\ddot{u}(x, t) = \frac{1}{r(x)}\operatorname{div}\Big\{g(x, 0)\nabla u(x, t)-\theta(x, t)l(x, 0)+\int_0^\infty [g'(x, s)\nabla u(x, t-s)-l'(x, s)$$

(1)
$$\times\theta(x, t-s)]ds\Big\}+f(x, t) \stackrel{\text{df}}{=} L(\nabla u, \theta)+f,$$

$$\dot{\theta}(x, t) = \theta_0/r(x)c(x, 0)\Big\{\operatorname{div}[k(x)\nabla\theta(x, t)/\theta_0]-l(x, 0)\nabla\dot{u}(x, t)+\int_0^\infty [l'(x, s)$$

$$\times\nabla\dot{u}(x, t-s)+r(x)c'(x, s)\dot{\theta}(x, t-s)/\theta_0]ds\Big\} \stackrel{\text{df}}{=} M(\nabla u, \theta).$$

Moreover, we assume the boundary conditions

(2)
$$u(x, t) = U(x, t) \quad \text{for} \quad (x, t)\in\partial B\times]0, T[\text{ (1)}$$
$$\theta(x, t) = Q(x, t) \quad \text{for} \quad (x, t)\in\partial B\times]0, T[$$

and "history conditions"

(3)
$$u(x, -s) = \bar{W}(x, s) \quad s \geqslant 0, \quad x\in\bar{B},$$
$$\theta(x, -s) = \bar{A}(x, s), \quad s \geqslant 0, \quad x\in\bar{B},$$

where the functions $U, Q, \bar{W}, \bar{A}$ are given.

PROBLEM 1: to find two curves

$$(-\infty, T]\ni t \stackrel{\varphi(t)}{\to} u(\cdot, t)\in H^1(B), \quad \varphi\in C^2\big((-\infty, T); H^1(B)\big),$$
$$(-\infty; T]\ni t \stackrel{\psi(t)}{\to} \theta(\cdot, t)\in H^1(B), \quad \psi\in C^1\big((-\infty, T); H^1(B)\big)$$

such that Eqs. (1) and the conditions (2) and (3) hold. Using the trace theorem we obtain the problem (1) which is equivalent to the following problem: to find two curves

$$(-\infty, T]\ni t \stackrel{\varphi(t)}{\to} u(\cdot, t)\in H^1(B), \quad \varphi\in C^2\big((-\infty, T); H^1(B)\big),$$
$$(-\infty, T]\ni t \stackrel{\psi(t)}{\to} \theta(\cdot, t)\in H^1(B), \quad \psi\in C^1\big((-\infty, T): H^1(B)\big)$$

satisfying

$$\ddot{u}(x, t) = L\big(\nabla u(x, t), \theta(x, t)\big)+F(x, t),$$
$$\dot{\theta}(x, t) = M\big(\nabla u(x, t), \theta(x, t)\big)+G(x, t)$$

the homogeneous boundary conditions

$$u(x, t) = 0 \quad \text{for} \quad (x, t)\in\partial B\times]0, T[,$$
$$\theta(x, t) = 0 \quad \text{for} \quad (x, t)\in\partial B\times]0, T[$$

(1) for any finite T.

and "history" conditions

$$u(x, -s) = W(x, s), \quad s \geqslant 0, \quad x \in \bar{B},$$
$$\theta(x, -s) = A(x, s), \quad s \geqslant 0, \quad x \in \bar{B},$$

where the functions F, G, W, A are given.

We want to resolve our question to the initial value problem $\frac{d}{dt} x = Ax+f$, $x(0) = x_0$ in the Hilbert space. To do that we give some assumptions:

I functions $r(x)$, $k(x)$, $g(x, s)$, $l(x, s)$, $c(x, s)$ for all fixed $s \geqslant 0$ are Lebesgue measurable and essentially bounded on B,

II functions

$$\left\| \frac{\partial^i}{\partial s^i} g(s) \right\| \stackrel{\mathrm{df}}{=} \left\| \frac{\partial^i}{\partial s^i} g(\cdot, s) \right\|_{L^\infty(B)},$$
$$\left\| \frac{\partial^i}{\partial s^i} c(s) \right\| \stackrel{\mathrm{df}}{=} \left\| \frac{\partial^i}{\partial s^i} c(\cdot, s) \right\|_{L^\infty(B)}, \quad i = 1, 2$$

are continuous and integrable on $[0, \infty)$,

III $g(x, s) = g^T(x, s)$ for all $s \geqslant 0$ ($g^T(x, s)$ the transpose of $g(x, s)$),

IV $\operatorname{essinf} r(x) \geqslant r_0 > 0$,

V $\operatorname{essinf} c(x, 0) \geqslant c_0 > 0$,

VI $\exists\ g > 0$ such that

$$\int_B \nabla y(x) g(x, \infty) \nabla y(x) dV \geqslant g \|\nabla y\|^2_{L^2(B)} \quad \text{for all} \quad y \in C_0^\infty(B),$$

where $g(x, \infty)$ denotes $\lim_{s\to\infty} g(x, s)$,

VII $\exists\ K > 0$ such that

$$\int_B \nabla b(x) k(x) \nabla b(x) dV \geqslant K \|\nabla b\|^2_{L^2(B)} \quad \text{for all} \quad b \in C_0^\infty(B),$$

VIII there exists a function $g_2(s) \geqslant 0$ such that

$$\int_B \nabla y(x) g''(x, s) \nabla y(x) dV \geqslant g_2(s) \|\nabla y\|^2_{L^2(B)} \quad \text{for all} \quad y \in C_0^\infty(B),$$

IX there exists a function $c_2(s) \geqslant 0$ such that

$$-\int_B c''(x, s) b^2(x) dV \geqslant c_2(s) \|b\|^2_{L^2(B)} \quad \text{for all} \quad b \in C_0^\infty(B),$$

X a function $\|l'(s)\| \stackrel{\mathrm{df}}{=} \|l'(\cdot, s)\|_{L^\infty(B)}$ is integrable on $[0, \infty)$,

XI $\|l''(s)\| \leqslant (r_0/\theta_0)^{1/2} c_2(s)^{1/2} g_2(s)^{1/2}$ for all $s \geqslant 0$,

XII $F(\cdot, t) \in L^2(B)$, $G(\cdot, t) \in L^2(B)$ and the functions

$$]0, T[\ \ni t \stackrel{F(t)}{\to} F(\cdot, t) \in L^2 B,$$
$$]0, T[\ \ni t \stackrel{G(t)}{\to} G(\cdot, t) \in L^2 B$$

are of class C^1 (have continuous derivative of order one).

REMARK 1

a) Assumptions X, XI and the Schwarz inequality give

$$||l'(s)|| \leqslant (r_0/\theta_0)^{1/2} c_1(s)^{1/2} g_1(s)^{1/2},$$

where

$$c_1(s) = \int_S^\infty c_2(p)\,dp, \quad g_1(s) = \int_S^\infty g_2(p)\,dp,$$

b) definition of the function $c_1(s)$ and assumption VIII

$$-\int_B \nabla y(x) g'(x,s) \nabla y(x)\,dV \geqslant g_1(s) ||\nabla y||^2_{L^2(B)} \quad \text{for all} \quad y \in C_0^\infty(B),$$

similarly

$$\int_B c'(x,s) b^2(x)\,dV \geqslant c_1(s) ||b||^2_{L^2(B)} \quad \text{for all} \quad b \in C_0^\infty(B).$$

Let us denote

$$X_0 = C_0^\infty(B) \times C_0^\infty(B) \times C_0^\infty(B) \times C^\infty([0,\infty); H_0^1(B)) \times C^\infty([0,\infty); H_0^1(B))$$

and introduce the bilinear form for elements of X_0

$$\langle (u, v, \theta, w(s), a(s)), (\bar{u}, \bar{v}, \bar{\theta}, \bar{w}(s), \bar{a}(s)) \rangle = \int_B \left[\nabla u g(\infty) \nabla \bar{u} + r v \bar{v} + \frac{1}{\theta_0} r c(0) \theta \bar{\theta} \right] dV$$

$$- \int_B \int_0^\infty [\nabla u - \nabla w(s)] g'(s) [\nabla \bar{u} - \nabla \bar{w}(s)] + \bar{a}(s) l'(s) [\nabla u - \nabla w(s)] + a(s) l'(s) [\nabla \bar{u} - \nabla \bar{w}(s)]$$

$$+ (r/\theta_0 + 1) c'(s) a(s) \bar{a}(s)\,ds\,dV$$

(where dependence on x is omitted).

REMARK 2.

The remark given by C. NAVARRO in the paper [3] p. 19 does not justify the statement that the bilinear form

$$\langle (u, v, \theta, w, a), (\bar{u}, \bar{v}, \bar{\theta}, \bar{w}, \bar{a}) \rangle_1 = \langle (u, v, \theta, w, a), (\bar{u}, \bar{v}, \bar{\theta}, \bar{w}, \bar{a}) \rangle$$

$$- \int_B \int_0^\infty c'(s) a(s) \bar{a}(s)\,ds\,dV$$

is an inner product.

P r o o f. We show that there exists an element $z = (0, 0, 0, w, a) \in X_0$ such that $z \neq 0$ and $\langle z, z \rangle_1 = 0$.

Let us assume that there is an interval $I = [p, q] \subset [0, \infty)$, $p < q < +\infty$ such that for all $s \in I$

$$g'_{ijkl}(x,s) = \begin{cases} g > 0 & \text{for} \quad x \in A \subset B \text{ where } A \text{ is a subregion of } B \text{ such that } \bar{A} \subset B \\ & \quad \text{and } r(x) = r_0 \text{ for } x \in A, \\ 0 & \text{for} \quad x \in B \setminus A, \end{cases}$$

$$l'_{ij}(x,s) = \begin{cases} 1 > 0 & \text{for} \quad x \in A, \\ 0 & \text{for} \quad x \in B \setminus A \end{cases}$$

and

$$c'(x, s) = \begin{cases} c > 0, & x \in A, \\ 0, & x \in B \setminus A. \end{cases}$$

Let us take $w \in C_0^\infty([p, q]; H_0^1(B))$ such that $(\nabla w)_{ij}(x, s) = f(s)$ for $x \in A$ and similarly $a \in C_0^\infty([p, q]; H_0^1(B))$ such that $a(x, s) = g(s)$ for $x \in A$. We assume $f(s), g(s) > 0$ for $s \in]p, q[$ and there exists a constant $h > 0$ such that $g(s) = hf(s)$. Then

$$|\langle z, z\rangle_1| = \left|\int_A \int_I \nabla w(s) g'(s) \nabla w(s) - 2a(s) l'(s) \nabla w(s) - r_0/\theta_0 c'(s) a^2(s) ds dV\right|$$

$$= \left|\int_A \int_I g \sum_{ijkl} f(s) f(s) - 2hf(s) l \sum_{ij} f(s) - r_0/\theta_0 ch^2 f^2(s) ds dV\right.$$

$$= |A| \left|\int_I (81g - 18hl - r_0/\theta_0 ch^2) f^2(s) ds\right|.$$

Let

$$\langle z, z\rangle_1 = 0 \quad \text{then} \quad 81g - 18hl - r_0/\theta_0 ch^2 = 0, \quad \Delta = 3241^2 + 324r_0/\theta_0, \quad cg > 0.$$

Using the assumptions XI, VIII and IX it is easy to show that $l^2 \leqslant r_0/\theta_0\, cg$. This inequality implies $\sqrt{\Delta} \geqslant \sqrt{628}\, l = 18\sqrt{2}\, l$ so that $h = 9\theta_0\big(r_0(\sqrt{2}-1)\, l\big)$, $c > 0$ thus $w, a \neq \neq 0$ q.e.d.

Let us denote

$$X_1 = \{(u, v, \theta, w, a) \in X_0 : \langle (u, v, \theta, w, a), (u, v, \theta, w, a)\rangle < +\infty\}.$$

REMARK 3

the form $\langle , \rangle$ is an inner product in X_1.

Proof. It is obvious that the form $\langle , \rangle$ is bilinear and symmetric. We show that this form is nonnegative

$$\langle (u, v, \theta, w, a), (u, v, \theta, w, a)\rangle = \int_B \left[\nabla u g(\infty) \nabla u + rvv + \frac{1}{\theta_0} rc(0)\theta^2\right] dV$$

$$- \int_B \int_0^\infty \left\{[\nabla u - \nabla w(s)] g'(s) [\nabla u - \nabla w(s)] + 2a(s) l'(s) [\nabla u - \nabla w(s)] - \frac{1}{\theta_0} rc'(s) a^2(s)\right\} ds dV$$

$$+ \int_B \int_0^\infty c'(s) a^2(s) ds dV, \quad \text{since IV, V and VI,}$$

$$\int_B \left[\nabla u g(\infty) \nabla u + rvv + \frac{1}{\theta_0} rc(0)\theta^2\right] dV \geqslant g\|\nabla u\|_{L^2}^2 + r_0\|v\|_{L^2}^2 + \frac{1}{\theta_0} r_0 c_0 \|\theta\|_{L^2}^2$$

hence the first integral is nonnegative. Using the Remark 1, the assumption V and the Schwarz inequality second integral,

$$- \int_B \int_0^\infty \left\{[\nabla u - \nabla w(s)] g'(s) [\nabla u - \nabla w(s)] + 2a(s) l'(s) [\nabla u - \nabla w(s)] - \frac{1}{\theta_0} rc'(s) a^2(s)\right\} ds dV$$

$$\geqslant \int_0^\infty [g_1(s)^{1/2} \|\nabla u - \nabla w(s)\|_{L^2} - (r_0/\theta_0)^{1/2} c_1(s)^{1/2} \|a(s)\|_{L^2}]^2 ds \geqslant 0 \quad \text{is nonnegative.}$$

Obviously

$$\int_B \int_0^\infty c'(s) a^2(s)\, ds\, dV \geqslant \int_0^\infty c_1(s) \|a(s)\|^2_{L^2} ds \geqslant 0$$

so

$$\langle (u, v, \theta, w, a), (u, v, \theta, w, a) \rangle \geqslant 0.$$

Furthermore we assume that

$$\langle (u, v, \theta, w, a), (u, v, \theta, w, a) \rangle = 0,$$

then

$u = 0$, $v = 0$, $\theta = 0$ and $a(s) = 0$ for all $s \geqslant 0$ thus $w(s) = 0$ for all $s \geqslant 0$ q.e.d.

DEFINITION. $X = \underset{\langle \cdot, \cdot \rangle}{\text{com}}\, X_1$ *i.e. X is obtained as the completion of X_1 under the inner product $\langle \cdot, \cdot \rangle$ (obviously X is the Hilbert space).*

We define the operator A in the space X

$$A \begin{bmatrix} u \\ v \\ \theta \\ w \\ a \end{bmatrix} = \begin{bmatrix} v \\ \frac{1}{r} \operatorname{div} \left\{ g(0) \nabla u - \theta l(0) + \int_0^\infty [g'(s) \nabla w(s) - l'(s) a(s)]\, ds \right\} \\ \frac{\theta_0}{rc(0)} \left\{ -l'(0) \nabla u - l(0) \nabla v + \operatorname{div}\left[\frac{1}{\theta_0} k \nabla \theta\right] - \frac{1}{\theta_0} rc'(0)\theta - \int_0^\infty \left[l''(s) \nabla w(s) + \frac{1}{\theta_0'} rc''(s) a(s) \right] ds \right\} \\ -w(s) \\ -a'(s) \end{bmatrix}$$

with the domain

$$D(A) = \{(u, v, \theta, w, a) \in X : A(u, v, \theta, w, a) \in X \quad \text{and} \quad w(0) = u \quad a(0) = \theta\}.$$

REMARK 4.

Our problem is equivalent to the following evolutionary equation

$$\frac{d}{dt} \begin{bmatrix} u(t) \\ v(t) \\ \theta(t) \\ w(t) \\ a(t) \end{bmatrix} = A \begin{bmatrix} u(t) \\ v(t) \\ \theta(t) \\ w(t) \\ a(t) \end{bmatrix} + \begin{bmatrix} 0 \\ F(t) \\ G(t) \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} u(0) \\ v(0) \\ \theta(0) \\ w(0) \\ a(0) \end{bmatrix} = \begin{bmatrix} W(0) \\ \lim_{s \to \infty} W'(s) \\ A(0) \\ W \\ A \end{bmatrix}. \tag{4}$$

THEOREM. *The operator A is the generator of a C_0 semigroup.*

Proof.

LEMMA. Let Y be the Hilbert space with an inner product $\langle , \rangle$ and A operator in Y. A is the generator of a C_0 semigroup if and only if:

i) there exists a constant $\beta \in R$ such that $\langle Ay, y \rangle \leqslant \beta \langle y, y \rangle$ for all $y \in D(A)$,

ii) Y is the range of the operator $(\lambda - A)$ for $\lambda > \beta$,

iii) $D(A)$ is dense in Y;

proof of the Lemma: see HILLE, PHILLIPS [2].

We show that the assumptions of the lemma hold

i) Let $(u, v, \theta, w, a) \in D(A)$ then

$$\langle A(u, v, \theta, w, a), (u, v, \theta, w, a)\rangle = -1/2 \int_B \int_0^\infty [\nabla u - \nabla w(s)] g''(s) [\nabla u - \nabla w(s)] ds dV$$

$$+ \int_B \int_0^\infty [\theta - a(s)] l''(s) [\nabla u - \nabla w(s)] ds dV + 1/2 \int_B \int_0^\infty r/\theta_0 c''(s) [\theta - a(s)]^2 ds dV$$

$$- \int_B \int_0^\infty c'(s) \frac{d}{ds} [1/2 a(s)] ds dV - \frac{1}{\theta_0} \int_B \nabla\theta k \nabla\theta dV$$

$$\leqslant -1/2 \int_0^\infty \{g_2(s) ||\nabla u - \nabla w(s)||^2_{L^2} - 2||l''(s)|| \, ||\theta - a(s)||_{L^2} ||\nabla u - \nabla w(s)||_{L^2}$$

$$+ c_2(s) ||\theta - a(s)||^2_{L^2}\} ds - \frac{1}{\theta_0} K ||\nabla\theta||^2_{L^2} + 1/2 ||c'(0)|| \, ||\theta||^2_{L^2}$$

$$\leqslant -1/2 [g_2(s)^{1/2} ||\nabla u - \nabla w(s)||_{L^2} - c_2(s)^{1/2} ||\theta - a(s)||_{L^2}]^2 ds + 1/2 ||c'(0)|| \, ||\theta||^2_{L^2}$$

hence

$$\langle A(u, v, \theta, w, a), (u, v, \theta, w, a)\rangle \leqslant \beta \frac{1}{\theta_0} r_0 c_0 ||\theta||^2_{L^2} \leqslant \beta (u, v, \theta, w, a), (u, v, \theta, w, a)\rangle,$$

where

$$\beta = ||c'(0)|| \theta / 2 r_0 c_0 .$$

ii) Let $(\bar{u}, \bar{v}, \bar{\theta}, \bar{w}, \bar{a}) \in X$, we find the element $(u, v, \theta, w, a) \in D(A)$ such that

$$(\lambda - A)(u, v, \theta, w, a) = (\bar{u}, \bar{v}, \bar{\theta}, \bar{w}, \bar{a}) \quad \text{for all} \quad \lambda > \beta > 0.$$

From the last two equations we have

$$w(s) = e^{-\lambda s} \left\{u + \int_0^s e^{\lambda p} \bar{w}(p) dp\right\},$$

$$a(s) = e^{-\lambda s} \left\{\theta + \int_0^s e^{\lambda p} \bar{a}(p) dp\right\},$$

thus we obtain the following system:

$$A' \begin{bmatrix} u \\ \theta \end{bmatrix} = \begin{bmatrix} E & F \\ C & D \end{bmatrix} \begin{bmatrix} u \\ \theta \end{bmatrix} = \begin{bmatrix} h_1 \\ h_2 \end{bmatrix},$$

where

$$Eu = \lambda^2 u - \frac{1}{r} \operatorname{div} \left\{\left[g(0) + \int_0^\infty e^{-\lambda s} g'(s) ds\right] \nabla u\right\},$$

$$F\theta = \frac{1}{r}\operatorname{div}\left\{\left[l(0)+\int_0^\infty l'(s)e^{-\lambda s}ds\right]\theta\right\},$$

$$Cu = \frac{\theta_0}{rc(0)}\left[l(0)+\int_0^\infty l'(s)e^{-\lambda s}ds\right]\nabla u,$$

$$D\theta = \theta-\frac{\theta_0}{rc(0)}\left\{\frac{1}{\lambda}\operatorname{div}\left(\frac{1}{\theta_0}k\nabla\theta\right)-\frac{r}{\theta_0}\int_0^\infty c'(s)e^{-\lambda s}ds\,\theta\right\},$$

$$h_1 = \bar{v}+\lambda\bar{u}+\frac{1}{r}\operatorname{div}\left\{\int_0^\infty\left[\int_s^\infty e^{-\lambda p}g'(p)dp\right]e^{\lambda s}\nabla\bar{w}(s)ds-\int_0^\infty\left[\int_s^\infty e^{-\lambda p}l'(p)dp\right]e^{\lambda p}\bar{a}(s)ds\right\},$$

$$h_2 = \frac{1}{\lambda}\bar{\theta}+\frac{\theta_0}{rc(0)}\left\{\frac{1}{\lambda}l(0)\nabla\bar{u}+\frac{1}{\lambda}\int_0^\infty\left[l'(s)\nabla\bar{w}(s)+\frac{r}{\theta_0}c'(s)\bar{a}s\,ds\right.\right.$$

$$\left.\left.-\int_0^\infty\left[\int_s^\infty e^{-\lambda p}l'(p)dp\right]e^{\lambda s}\nabla\bar{w}(s)ds-\int_0^\infty\left[\int_s^\infty e^{-\lambda p}c'(p)dp\right]e^{\lambda s}\bar{a}(s)ds\right\}.$$

We show the existence of a unique pair $(u,\theta)\in H_0^1(B)\times H_0^1(B)$ such that $A'(u,\theta) = (h_1,h_2)$. Let us denote $\langle,\rangle_2$ an inner product in $L^2(B)\times L^2(B)$ with weights r and $\frac{1}{\theta_0}rc(0)$. Let us define the bilinear form

$$B[(u,\theta),(u',\theta')] \overset{\text{df}}{=} \langle A'(u,\theta),(u',\theta')\rangle_2.$$

This form is the inner product in $H_0^1(B)\times H_0^1(B)$ such that the norm induced by this inner product is equivalent to the norm $\|\cdot\|_{H_0^1}+\|\;\|_{H_0^1}$, since it is easy to see that there exist two constants $l, L>0$ such that

$$l(\|u\|_{H_0^1}+\|\theta\|_{H_0^1}) \leqslant B[(u,\theta),(u,\theta)] \leqslant L(\|u\|_{H_0^1}+\|\theta\|_{H_0^1})$$

for all $(u,\theta)\in H_0^1\times H_0^1$. Thus $H_0^1(B)\times H_0^1(B)$ with the inner product $B[(u,\theta),(u,\theta)]$ is the Hilbert space. Now we show that $(h_1,h_2)\in H^{-1}\times H^{-1}$ with the norm $\|\;\|_{H^{-1}}+\|\;\|_{H^{-1}}$.

Let $y\in H_0^1(B)$; then

$$\left|\int_B ryh_1\,dV\right| \leqslant \int_0^\infty\left\{\left[\int_B\nabla y\left(-\int_s^\infty g'(p)e^{-\lambda p}dp\right)e^{\lambda s}\nabla y\,dV\right]^{1/2}\cdot\left[\int_B\nabla\bar{w}(s)\left(-\int_t^\infty g'(p)e^{-\lambda p}dp\right)\right.\right.$$

$$\left.\left.\times e^{\lambda s}\nabla\bar{w}(s)dV\right]^{1/2}+\left\|\int_s^\infty l'(p)e^{-\lambda p}dp\right\|e^{\lambda s}\|\nabla y\|_{L^2}\|a(s)\|_{L^2}\right\}ds+\|\bar{v}+\lambda\bar{u}\|_{L^2}\|y\|_{H_0^1},$$

note that

$$\int_B\nabla y\int_s^\infty g'(p)e^{-\lambda p}dp\nabla y\,dV \geqslant \frac{1}{\lambda}\int_B\nabla y g'(s)\nabla y\,dV$$

and

$$e^{\lambda s}\left\|\int_s^\infty l'(p)e^{-\lambda p}dp\right\| \leqslant \frac{1}{\lambda}\left[\frac{r_0}{\theta_0}c_1(s)g_1(s)\right]^{1/2}$$

since $||g'(s)||$ is integrable and $(\bar{u}, \bar{v}, \bar{\theta}, \bar{w}, \bar{a}) \in X$ $\left|\int\limits_B ryh_1 dV\right| \leqslant C||y||_{H_0^1}$ (C—constant) we obtain $h \in H^{-1}(B)$. Similarly it can be shown that $h_2 \in H^{-1}(B)$. Thus using the Riesz's theorem we have the result that there exists the unique pair $(u, \theta) \in H_0^1(B) \times H_0^1(B)$ such that

$$\langle (h_1, h_2), (u', \theta')\rangle_2 = B[(u, \theta), (u', \theta')] = \langle A'(u, \theta), (u', \theta')\rangle_2 \quad \text{for all} \quad (u', \theta') \in H_0^1 \times H_0^1$$

hence

$$A'(u, \theta) = (h_1, h_2).$$

It should be shown that $(u, v, \theta, w, a) \in D(A)$. From the definition $w(0) = u$ and $a(0) = \theta$; moreover, the element $(u, v, \theta, 0, 0) \in X$ since $u, v, \theta \in H_0^1$. We show that the element $(0, 0, 0, w, a) \in X$. Note that

$$-\int\limits_B \int\limits_0^\infty \nabla w(s) g'(s) \nabla w(s) \leqslant -\frac{1}{\lambda^2} \int\limits_B \int\limits_0^\infty \nabla \bar{w}(s) g'(s) \nabla \bar{w}(s) ds dV + \frac{1}{\lambda} \int\limits_B -\nabla u g'(0) \nabla u dV < +\infty,$$

$$\int\limits_B \int\limits_0^\infty \left(\frac{r}{\theta_0}+1\right) c'(s) a^2(s) ds dV \leqslant \int\limits_B \int\limits_0^\infty \left(\frac{r}{\theta_0}+1\right) c'(s) \bar{a}^2(s) ds dV + \frac{1}{\lambda} \int\limits_B \left(\frac{r}{\theta_0}+1\right) c'(0) \theta^2 dV < +\infty,$$

and

$$\left|\int\limits_B \int\limits_0^\infty a(s) l'(s) \nabla w(s) ds dV\right| \leqslant \int\limits_B \int\limits_0^\infty \left[\left(\frac{r}{\theta_0}+1\right) c'(s) a^2(s) - \nabla w(s) g'(s) \nabla w(s)\right] ds dV < +\infty$$

thus $||0, 0, 0, w, a|| < \infty$. The last inequality and $(\lambda - A)(u, v, \theta, w, a) = (\bar{u}, \bar{v}, \bar{\theta}, \bar{w}, \bar{a})$ implies that $(u, v, \theta, w, a) \in D(A)$,

iii) the set

$$Y = \{(u, v, \theta, w, a) \in C_0^\infty(B) \times C_0^\infty(B) \times C_0^\infty(B) \times C_0^\infty([0, \infty); H_0^1(B)) \times C_0^\infty([0, \infty); H_0^1(B)) : w(0) = u \quad \text{and} \quad a(0) = \theta\}$$

is dence in X and obviously $Y \subset D(A)$

q.e.d.

LEMMA. Let X be the Banach space, A the operator in X satisfying the assumptions of the Hille Yosida theorem and additionally a function $f: [0, T] \to X$ of class C^1 and $x_0 \in D(A)$; then there exists a unique curve $x(t)$: $[0, T] \to X$ of class C^1 such that

$$\frac{d}{dt} x(t) = Ax(t) + f(t) \quad \text{and} \quad x(0) = x_0.$$

This function is given by the formula

$$x(t) = U(t)x_0 + \int_0^t U(t-s)f(s)ds,$$

where $U(t)$ is the semigroup generated by A.

Proof: see T. KATO [5].

COROLLARY. There exists the unique solution of Eq. (4).

Proof: from the last lemma and the theorem.

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