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On generalized solutions of a nonlinear boundary value problem of elasticity
Nonresonance case

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THE DIRICHLET problem for the general Lamé equations is considered. The right hand side of Lamé equations depends on the displacement vector function. Sufficient conditions are given for the existence of the weak solution to the problem under the assumption that the homogeneous problem has only the trivial weak solution (nonresonance case). To prove existency Schauder's Fixed Point Theorem is employed.

Rozważono zagadnienie Dirichleta dla ogólnego układu równań Lamégo. Prawa strona równań Lamégo zależy od wektorowej funkcji przemieszczenia. Podano warunki dostateczne dla istnienia słabych rozwiązań problemu przy założeniu, że zagadnienie jednorodne posiada wyłącznie trywialne rozwiązanie słabe (przypadek bezrezonansowy). Dla udowodnienia twierdzenia o istnieniu rozwiązania zastosowano twierdzenie Schaudera.

Рассмотрена задача Дирихле для общей системы уравнений Ламе. Правая сторона уравнений Ламе зависит от векторной функции перемещения. Приведены достаточные условия для существования слабых решений задачи при предположении, что однородная задача обладает исключительно тривиальным слабым решением (безрезонансный случай). Для доказательства теоремы существования решения применена теорема Шаудера.

1. Introduction

LET $D \subset R^3$ be a domain with a Lipschitz boundary ∂D. We consider the following Dirichlet Problem of the Theory of Elasticity. Find the displacement u that satisfies the general Lamé equations

$$(1.1)\quad \frac{\partial}{\partial x_i}(\lambda \operatorname{div} u)+\sum_{j=1}^{3}\left[\frac{\partial}{\partial x_j}\,\mu\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right)\right]+\omega^2 u_i = -g_i\big(x, u(x)\big), \qquad x \in D,$$

subject to the conditions

$$(1.2)\qquad u_i(x) = 0, \quad x \in \partial D \quad (i = 1, 2, 3),$$

where λ, μ are Lamé coefficients, ω is a real constant, $g\colon D \times R^3 \to R^3$ is a given vector-function.

The problem (1.1), (1.2) in the linear case, i.e. $g\big(x, u(x)\big) \equiv g(x)$, has been considered by many authors. The classical solution has been obtained, among others, by V. D. KUPRADZE in his monograph [4]. J. NEČAS and T. HLAVÁČEK [5] proved the existence of a generalized solution by the Variational Method.

A nonlinear problem of a somewhat more complicated type, but only in the case of constant Lamé coefficients, has been considered in the paper [2]. In this paper we are concerned with the existence for weak solutions of the problem (1.1), (1.2).

We introduce the following spaces:

$$W = [L^2(D)]^3, \quad (u \in W \text{ means } u_i \in L^2(D), \quad i = 1, 2, 3),$$

$$\mathring{W} = [\mathring{H}_1]^3, \quad \mathring{H}_1 \text{ is the usual Sobolev space.}$$

The norms in W and $\mathring{W}$ are given by

$$||u||_W = \Big(\sum_{i=1}^{3} ||u_i||^2_{L^2}\Big)^{1/2}, \quad \text{where} \quad ||u_i||_{L^2} = \Big(\int_D |u_i|^2 dx\Big)^{1/2},$$

$$||u||_{\mathring{W}} = \Big(\sum_{i=1}^{3} ||u_i||^2_{\mathring{H}_1}\Big)^{1/2}, \quad \text{where} \quad ||u_i||_{\mathring{H}_1} = \Big(\int_D (|\nabla u_i|^2 + |u_i|^2) dx\Big)^{1/2}.$$

Spaces W, $\mathring{W}$ are Hilbert spaces. Inner products have the following form:

$$(u, v)_W = \sum_{i=1}^{3} (u_i, v_i)_{L^2} = \int_D \Big(\sum_{i=1}^{3} u_i v_i\Big) dx,$$

$$(u, v)_{\mathring{W}} = \sum_{i=1}^{3} (u_i, v_i)_{\mathring{H}_1} = \int_D \sum_{i=1}^{3} (\nabla u_i \nabla v_i + u_i v_i) dx,$$

respectively.

Assume that the functions $g_i(x, u_1, u_2, u_3)$, $i = 1, 2, 3$, satisfy the Carathéodory continuity condition, i.e. g_i are measurable with respect to x for fixed (u_1, u_2, u_3), $u_i \in (-\infty, \infty)$ and continuous in (u_1, u_2, u_3) for almost all $x \in D$.

The Lamé coefficients satisfy the following conditions: λ, $\mu \in L^\infty(D)$ and $\lambda(x) \geqslant 0$, $\mu(x) \geqslant \mu_0 > 0$ for all $x \in \bar{D}$, where μ_0 is a constant.

DEFINITION 1.1. *A function u is said to be the weak generalized solution to the Dirichlet problem* (1.1), (1.2), *if*

$$u \in \mathring{W}, \tag{1.3}$$

$$\int_D \Big[\lambda(\operatorname{div} u)(\operatorname{div} v) + \frac{\mu}{2} \sum_{i,j=1}^{3} \Big(\frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j}\Big)\Big(\frac{\partial v_j}{\partial x_i} + \frac{\partial v_i}{\partial x_j}\Big)\Big] dx$$
$$-\omega^2 \int_D \Big(\sum_{i=1}^{3} u_i v_i\Big) dx = \int_D \Big(\sum_{i=1}^{3} g_i(x, u(x)) v_i(x)\Big) dx \quad \textit{for all} \quad v \in \mathring{W}. \tag{1.4}$$

The meaning of the definition of the weak solution is as follows. The condition (1.3) means that $u = 0$ on ∂D in the sense of traces. Suppose now that the weak solution u is sufficiently smooth. By Green's formula [5 p. 20] we have from Eq. (1.4)

$$\int_D \Big\{\frac{\partial}{\partial x_i}(\lambda \operatorname{div} u) + \sum_{j=1}^{3} \Big[\frac{\partial}{\partial x_j} \mu \Big(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\Big)\Big] + \omega^2 u_i + g_i(x, u(x))\Big\} v_i dx = 0, \tag{1.5}$$

for all $v = \mathring{W}$, $(i = 1, 2, 3)$.

This means that u satisfies Lamé equations (1.1) in the weak sense; they are met almost everywhere in D in case the expression in the brackets belongs to $L^2(D)$.

2. Preliminaries

Let us define in $\mathring{W}$ a bilinear form $\langle u, v\rangle_{\mathring{W}}$ by the following formula:

$$(2.1)\qquad \langle u, v\rangle_{\mathring{W}} = \int_D \left[\lambda(\operatorname{div} u)(\operatorname{div} v)+\frac{\mu}{2}\sum_{i,j=1}^{3}\left(\frac{\partial u_j}{\partial x_i}+\frac{\partial u_i}{\partial x_j}\right)\left(\frac{\partial v_j}{\partial x_i}+\frac{\partial v_i}{\partial x_j}\right)\right]dx.$$

The bilinear form $\langle u, v\rangle_{\mathring{W}}$ satisfies the axioms of the inner product and defines a norm on $\mathring{W}$ by

$$(2.2)\qquad ||u||_{\tilde{\mathring{W}}} = (\langle u, u\rangle_{\mathring{W}})^{1/2}.$$

The norm $||u||_{\tilde{\mathring{W}}}$ is equivalent to the usual norm in $\mathring{W}$, i.e. there exist positive constants c_1, c_2 such that for all $u \in \mathring{W}$

$$(2.3)\qquad ||u||_{\tilde{\mathring{W}}} \leqslant c_1||u||_{\mathring{W}},$$

$$(2.4)\qquad c_2||u||_{\mathring{W}} \leqslant ||u||_{\tilde{\mathring{W}}}.$$

In fact, by the assumptions on λ and μ

$$(||u|_{\tilde{\mathring{W}}})^2 = \int_D \left[\lambda(\operatorname{div} u)^2+\frac{\mu}{2}\sum_{i,j=1}^{3}\left(\frac{\partial u_j}{\partial x_i}+\frac{\partial u_i}{\partial x_j}\right)^2\right]dx$$

$$\leqslant \int_D \left[3\lambda\sum_{i=1}^{3}\left(\frac{\partial u_i}{\partial x_i}\right)^2+\mu\sum_{i,j=1}^{3}\left(\left(\frac{\partial u_j}{\partial x_i}\right)^2+\left(\frac{\partial u_i}{\partial x_j}\right)^2\right)\right]dx$$

$$\leqslant 3||\lambda||_\infty\sum_{i=1}^{3}||u_i||^2_{\mathring{H}_1}+2||\mu||_\infty\sum_{i=1}^{3}||u_j||^2_{\mathring{H}_1} = (3||\lambda||_\infty+2||\mu||_\infty)||u||^2_{\mathring{W}}.$$

Denoting $(3||\lambda||_\infty+2||\mu||_\infty)^{1/2} = c_1$, we get the inequality (2.3).

To prove the inequality (2.4) we take advantage of the following Korn's inequality ([3] section 12):

$$(2.5)\qquad \int_D \sum_{i,j=1}^{3}\left[\frac{1}{2}\left(\frac{\partial u_j}{\partial x_i}+\frac{\partial u_i}{\partial x_j}\right)\right]^2 dx \geqslant C||u||^2_{\mathring{W}},$$

for $u \in \mathring{W}$, (C is a constant).

We have now

$$(||u||_{\tilde{\mathring{W}}})^2 \geqslant 2\int_D \mu\sum_{i,j=1}^{3}\left[\frac{1}{2}\left(\frac{\partial u_j}{\partial x_i}+\frac{\partial u_i}{\partial x_j}\right)\right]^2 dx$$

$$\geqslant 2\mu_0\int_D \sum_{i,j=1}^{3}\left[\frac{1}{2}\left(\frac{\partial u_j}{\partial x_i}+\frac{\partial u_i}{\partial x_j}\right)\right]^2 dx \geqslant 2\mu_0 C||u||^2_{\mathring{W}}.$$

Taking $c_2 = \sqrt{2\mu_0 C}$, we obtain the inequality (2.4).

An inner product $(u, v)_W$ for a fixed $u \in W$ is a linear continuous functional on $\mathring{W}$ with respect to the norm $||\cdot||_{\tilde{\mathring{W}}}$. Indeed

$$|(u, v)_W| = \left|\sum_{i=1}^{3} \int_D u_i v_i dx\right| \leqslant \sum_{i=1}^{3} ||u_i||_{L^2} ||v_i||_{L^2}$$

$$\leqslant ||u||_W ||v||_W \leqslant ||u||_W ||v||_{\mathring{W}} \leqslant \frac{1}{c_2} ||u||_W ||v||_{\tilde{\mathring{W}}}.$$

By Riesz's Theorem there exists a unique element $Au \in \mathring{W}$ such that

$$\langle Au, v\rangle_{\mathring{W}} = (u, v)_W \quad \text{for all} \quad v \in \mathring{W}. \tag{2.6}$$

The element Au satisfies the inequality

$$||Au||_{\tilde{\mathring{W}}} \leqslant \frac{1}{c_2} ||u||_W. \tag{2.7}$$

The relation (2.6) defines an operator $A: W \to \mathring{W}$ being linear and continuous on W. The restriction of the operator A to $\mathring{W}$, i.e. $A|_{\mathring{W}}: \mathring{W} \to \mathring{W}$ is compact. This is a direct corollary from the Rellich's Theorem (see [1] p. 30).

3. The sufficient condition

THEOREM 3.1. *Let $g = [g_1, g_2, g_3]$ be the vector-function satisfying conditions as specified in Sect.* 1. *Assume further, that g_i $(i = 1, 2, 3)$ satisfy the following inequality*:

$$|g_i(x, u_1, u_2, u_3)| \leqslant a_i(x) + b \sum_{k=1}^{3} |u_k|, \tag{3.1}$$

where $a_i(x) \in L^2(D)$, $b > 0$.

If the linear homogeneous problem

$$\frac{\partial}{\partial x_i} (\lambda \operatorname{div} u) + \sum_{j=1}^{3} \left[\frac{\partial}{\partial x_j} \mu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)\right] + \omega^2 u_i = 0, \qquad x \in D, \tag{3.2}$$

$$u_i(x) = 0, \qquad x \in \partial D \qquad (i = 1, 2, 3), \tag{3.3}$$

has only the trivial solution, then there exists a weak solution of the nonlinear problem (1.1), (1.2).

Proof. For fixed $u \in W$ and arbitrary $v \in \mathring{W}$ we have

$$\left|\left(g(x, u(x)), v\right)_W\right| = \left|\sum_{i=1}^{3} \int_D g_i(x, u(x)) v_i(x) dx\right|$$

$$\leqslant \sum_{i=1}^{3} \left(||g_i(x, u(x))||_{L^2} ||v_i||_{L^2}\right) \leqslant (M_1 + M_2 + M_3) ||v||_{\mathring{W}} \leqslant \frac{M_1 + M_2 + M_3}{c_2} ||v||_{\tilde{\mathring{W}}},$$

where $||g_i(x, u(x))||_{L^2} = M_i$ for $i = 1, 2, 3$, (M_i may depend on u). Hence the inner product $\big(g(x, u(x)), v\big)_W$ is the linear continuous functional on $\mathring{W}$ with respect to the norm $||\cdot||_{\mathring{W}}^{\sim}$. Therefore, there exists a unique $T(u) \in \mathring{W}$ such that

$$\langle T(u), v\rangle_{\mathring{W}} = \big(g(x, u(x)), v\big)_W, \tag{3.4}$$

while

$$||T(u)||_{\mathring{W}}^{\sim} \leqslant \frac{M_1+M_2+M_3}{c_2}. \tag{3.5}$$

Operator T: $W \to \mathring{W}$ is not, in general, linear but, as we shall prove, continuous. In fact, for all $u, v \in W$

$$\begin{aligned}(||T(u)-T(v)||_{\mathring{W}}^{\sim})^2 &= \langle T(u)-T(v), T(u)-T(v)\rangle_{\mathring{W}} \\ &= \big(g(x, u)-g(x, v), T(u)-T(v)\big)_W \leqslant ||g(x, u)-g(x, v)||_W\, ||T(u)-T(v)||_W \\ &\leqslant \frac{1}{c_2}\, ||g(x, u)-g(x, v)||_W\, ||T(u)-T(v)||_{\mathring{W}}^{\sim}.\end{aligned}$$

Hence

$$\begin{aligned}||T(u)-T(v)||_{\mathring{W}}^{\sim} &\leqslant \frac{1}{c_2}\, ||g(x, u)-g(x, v)||_W \\ &= \frac{1}{c_2}\Big(\sum_{i=1}^{3}\int_D |g_i\big(x, u_1(x), u_2(x), u_3(x)\big)-g_i\big(x, v_1(x), v_2(x), v_2(x)\big)|^2 dx\Big)^{1/2}.\end{aligned} \tag{3.6}$$

By the conditions imposed on the $g_i\big(x, u_1(x), u_2(x), u_3(x)\big)$ operators G_i: $W \to W$ defined by $G_i u = g_i\big(x, u_1(x), u_2(x), u_3(x)\big)$ are continuous (see Theorem 19.2 of [6]). Hence the inequality (3.6) implies continuity of the operator T. By Rellich's Theorem the restriction $T|_{\mathring{W}}$: $\mathring{W} \to \mathring{W}$ is compact.

In view of the definition (2.1), the relations (2.6) and (3.4), Eq. (1.4) may be written in a purely operator form:

$$u-\omega^2 Au = T(u), \quad u \in \mathring{W}. \tag{3.7}$$

Since the only solution of the homogeneous equation $(I-\omega^2 A)u = 0$ is the trivial solution $u = 0$, by Fredholm's Alternative (operator A is linear and compact) it follows that there exists a continuous inverse operator $(I-\omega^2 A)^{-1}$. Now Eq. (3.7) takes the form

$$u = (I-\omega^2 A)^{-1} T(u), \quad u \in \mathring{W}. \tag{3.8}$$

Denote $\Phi = (I-\omega^2 A)^{-1} T$, Φ: $\mathring{W} \to \mathring{W}$. The operator Φ is compact since it is the composition of continuous and compact operators.

By using Schauder's Fixed Point Theorem: "A compact operator Φ of a closed bounded convex set K in a Banach space X into itself has a fixed point", we will prove that Φ has a fixed point in $\mathring{W}$ — a solution of Eq. (3.8), i.e. a weak solution of the nonlinear problem (1.1), (1.2).

Let us consider a ball $K(0, R) = \{u \in \mathring{W};\ ||u||_{\mathring{W}}^{\sim} \leqslant R\}$, where $R > 0$ is a constant which will be chosen later on. $K(0, R)$ is a closed bounded convex set in the Banach space $\mathring{W}$.

Denote $||(I-\omega^2 A)^{-1}|| = N$. By the definition of the operator Φ we have

$$||\Phi(u)||_{\mathring{W}}^{\sim} \leqslant N||T(u)||_{\mathring{W}}^{\sim}. \tag{3.9}$$

To prove that Φ maps $K(0, R)$ into itself, we have to find estimates of M_i, $(i = 1, 2, 3)$. By the assumption (3.1)

$$M_i^2 = \int_D |g_i(x, u(x))|^2 dx \leqslant \int_D \Big[a_i^2(x) + 2a_i(x) b \sum_{k=1}^{3} |u_k| + b^2 \Big(\sum_{k=1}^{3} |u_k|\Big)^2\Big] dx$$

$$\leqslant ||a_i||_{L^2}^2 + 2\sqrt{3}\, b ||a_i||_{L^2} ||u||_W + 3b^2 ||u||_W^2 \leqslant ||a_i||_{L^2}^2 + 2\sqrt{3}\, \frac{b}{c_2} ||a_i||_{L^2} ||u||_{\tilde{\mathring{W}}} + 3\frac{b^2}{c_2^2} ||u||_{\tilde{\mathring{W}}}^2.$$

Taking advantage of Eqs. (3.5) and (3.9) we obtain

$$(3.10)\quad ||\Phi(u)||_{\tilde{\mathring{W}}} \leqslant \frac{\sqrt{3}\, N}{c_2} \left[\sum_{i=1}^{3} \left(||a_i||_{L^2}^2 + 3\sqrt{3}\, \frac{b}{c_2} ||a_i||_{L^2} ||u||_{\tilde{\mathring{W}}} + \frac{3b^2}{c_2^2} ||u||_{\tilde{\mathring{W}}}^2\right)\right]^{1/2}.$$

Let $u \in K(0, R)$. From Ineq. (3.10) it follows that if the inequality

$$(3.11)\quad \frac{3N^2}{c_2^2} \sum_{=}^{3} \left(||a_i||_{L^2}^2 + 3\sqrt{3}\, \frac{b}{c_2} ||a_i||_{L^2} R + \frac{3b^2}{c_2^2} R^2\right) \leqslant R^2$$

is satisfied, then $\Phi(u) \in K(0, R)$.

The inequality (3.11) is satisfied for

$$R \geqslant \frac{A_2 + \sqrt{4A_1(1 - A_3) + A_2^2}}{2(1 - A_3)},$$

where

$$A_1 = \frac{3N^2}{c_2^2} \sum_{i=1}^{3} ||a_i||_{L^2}^2, \qquad A_2 = \frac{9\sqrt{3}\, N^2 b}{c_2^3} \sum_{i=1}^{3} ||a_i||_{L^2},$$

$$A_3 = \frac{27N^2 b^2}{c_2^4}, \quad \text{provided} \quad 0 < b < \frac{c_2^2}{3\sqrt{3}\, N}.$$

Therefore by Schauder's Fixed Point Theorem Φ has a fixed point in $K(0, R) \subset \mathring{W}$.

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