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Surface waves in elastic Hadamard material

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IN THE PRESFNT paper we study the existence and properties of Rayleigh waves propagating in homogeneous isotropic Hadamard materials. An infinitesimal deformation is superposed on finite homogeneous static deformation. The constitutive functions must satisfy some restrictions, first of all because we demand that acceleration waves may propagate through the medium. It is shown that for some situations there exists no Rayleigh type wave, and this result is in accordance with the conditions under which the surface becomes unstable. Finally, we discuss two different models of Hadamard material in order to show how the initial deformation and material constants have an effect on the propagation of Rayleigh wave. This knowledge could be useful in determining the forms of the strain energy function.

Praca poświęcona jest analizie fal Rayleigha propagujących się w izotropowym materiale sprężystym typu Hadamarda. Na statyczne, jednorodne i skończone deformacje półprzestrzeni zostały nałożone odkształcenia nieskończenie małe. Wyprowadzono warunki ograniczające na funkcje konstytutywne wynikające z żądania aby w półprzestrzeni propagowały się fale przyspieszenia. Udowodniono, że istnienie fali Rayleigha zależy od funkcji materiałowych i od wstępnych dużych deformacji. Nieistnienie fali Rayleigha jest równoważne utracie stateczności półprzestrzeni. Dla dwóch modeli materiału typu Hadamarda wyprowadzono warunki istnienia fali Rayleigha. Warunki te mogą być pomocne przy określaniu funkcji energii.

Работа посвящена анализу релеевских волн, распространяющихся в изотропном упругом материале типа Адамара. На статические, однородные и конечные деформации полупространства наложены бесконечно малые деформации. Выведены ограничивающие условия для определяющих функций, вытекающие из требования, чтобы в полупространстве распространялись волны ускорения. Доказано, что существование релеевской волны зависит от материальных функций и от предварительных больших деформаций. Несуществование релеевской волны эквивалентно потери устойчивости полупространства. Для двух моделей материала типа Адамара выведены условия существования релеевской волны. Эти условия могут быть полезными при определении функции энергии.

1. Introduction

IN THE PRESENT paper we study the existence and properties of Rayleigh waves propagating in homogeneous isotropic Hadamard materials. As a basis we use the results obtained by HAYES, RIVLIN [2] for isotropic hyperelastic materials in cases when an infinitesimal deformation is superposed on finite homogeneous static deformation. The governing system of equations is linear. The coefficients depend on material properties and on the amount of the initial deformation. These coefficients must satisfy some restrictions, first of all because we demand that acceleration waves may propagate through the medium and secondly from the vanishing of the traction on the plane bounding surface.

The next step is to examine whenever Rayleigh waves can propagate. If the initial deformation is suitable, we show that for an arbitrary Hadamard material there is only one Rayleigh wave and it is a retrograde. For some values of the initial deformation and of the material constants it is shown that there exists no Rayleigh type wave propagating parallel to the free surface of the body and with its amplitude decreasing with distance from the surface. This result is an accordance with the conditions obtained by USMANI, BEATTY [4] under which the surface becomes unstable.

Finally we discuss two different models of Hadamard material in order to show how the initial deformation and material constants have an effect on the propagation of the Rayleigh wave. This knowledge could be useful in determining the forms of the strain energy function for Hadamard material.

2. The system of equations

First we investigate the full system of equations of motion for an isotropic hyperelastic half-space occupying the region $X_2 \geqslant 0$. We assume that motion depends only on (X_1, X_2, t) and we have a decomposition:

$$\bar{x}_l = x_l + \varepsilon u_l. \tag{2.1}$$

It means that we can write the deformation $\bar{F}_{lL} = \partial\bar{x}_l/\partial X_L$ as

$$\bar{F}_{lL} = F_{lL} + \varepsilon u_{l,L}. \tag{2.2}$$

Let us assume that the initial deformation is static and pure homogeneous:

$$\mathbf{F} = \begin{pmatrix} \lambda_2 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}.$$

The Cauchy stress tensor σ_{ij} associated with the doformation (2.3) is given by

$$\begin{aligned} \sigma_{\alpha\beta} &= 0 \qquad (\alpha \neq \beta), \\ \sigma_{\alpha\alpha} &= 2\mathrm{III}^{-1/2}\{\lambda_\alpha^2 W_1 + \lambda_\alpha^2(\mathrm{I} - \lambda_\alpha^2) W_2 + \mathrm{III}\, W_3\}, \end{aligned} \tag{2.4}$$

where

$$\mathrm{I} = \lambda_1^2 + \lambda_2^2 + \lambda_3^2, \quad \mathrm{II} = \lambda_1^2\lambda_2^2 + \lambda_2^2\lambda_3^2 + \lambda_1^2\lambda_1^2, \quad \mathrm{III} = \lambda_1^2\lambda_2^2\lambda_3^2,$$
$$W_1 = \partial W/\partial \mathrm{I}, \quad W_2 = \partial W/\partial \mathrm{II}, \quad W_3 = \partial W/\partial \mathrm{III}$$

and W is the strain energy function $W = W(\mathrm{I}, \mathrm{II}, \mathrm{III})$. The incremental stress $\bar{\sigma}_{ij}$ associated with the deformation (2.2) must satisfy the equation of motion in the coordinate system $\mathbf{x}$:

$$\bar{\sigma}_{ij,j} = \varrho \dot{v}_i, \tag{2.5}$$

where

$$v_i = \dot{u}_i \equiv u_{i,t}.$$

For the half-space $X_2 = 0$ we assume that the displacement $\mathbf{u}$ is simplified to

$$\begin{aligned} u_1 &= u_1(x_1, x_2, t), \\ u_2 &= u_2(x_1, x_2, t), \\ u_3 &= 0 \end{aligned}$$

and thus the stress $\bar{\sigma}_{ij}$ has the components (cf. HAYES, RIVLIN [1, 2]):

$$(2.7)\quad \begin{aligned} \bar{\sigma}_{11} &= c_{11}H_{11}+c_{12}H_{22}, \\ \bar{\sigma}_{22} &= c_{21}H_{11}+c_{22}GH_{22}, \\ \bar{\sigma}_{12} &= \bar{\sigma}_{21} = 2\mathrm{III}^{-1/2}\{W_1+\lambda_3^2W_2\}(\lambda_1^2H_{21}+\lambda_2^2H_{12}) \equiv b(\lambda_1^2H_{21}+\lambda_2^2H_{12}), \\ \bar{\sigma}_{33} &= \bar{\sigma}_{31} = \bar{\sigma}_{13} = \bar{\sigma}_{23} = \bar{\sigma}_{32} = 0, \end{aligned}$$

where

$$b \equiv 2\mathrm{III}^{-1/2}(W_1+\lambda_3^2W_2), \qquad H_{ij} \equiv u_{i,j}, \qquad \mathbf{H} = \begin{pmatrix} H_{11} & H_{12} & 0 \\ H_{21} & H_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Finally the governing system of equations contains the equation of motion and compatability conditions:

$$\varrho\dot{v}_i - \frac{\partial\bar{\sigma}_{ij}}{\partial H_{lk}}H_{lk,j} = 0,$$
$$\dot{H}_{ij}-v_{i,j} = 0$$

leading to

$$(2.9)\quad \begin{aligned} &\dot{v}_1 - \frac{1}{\varrho}(c_{11}H_{11,1}+b\lambda_2^2H_{12,2}+b\lambda_1^2H_{21,2}+C_{12}H_{22,1}) = 0, \\ &\dot{v}_2 - \frac{1}{\varrho}(c_{21}H_{11,2}+b\lambda_2^2H_{12,1}+b\lambda_1^2H_{21,1}+c_{22}H_{22,2}) = 0, \\ &\dot{H}_{ij}-v_{i,j} = 0, \quad i,j = 1,2. \end{aligned}$$

We are interested in the material in which the acceleration (plane) waves can propagate. Under this regime the constitutive function (energy function) must satisfy some restrictions. The general three-dimensional case was investigated in several papers (HAYES, RIVLIN [1], OGDEN [3]). The discussion of this problem for the half-space (system (2.9)) is simpler, it means that the well-known result is possible to get very quickly. The assumption that the bounding surface $X_2 = 0$ is traction-free in the configuration leads to additional restrictions for λ_1, λ_2, λ_3 and the strain energy function W. Then let us recall very briefly. We call a surface Σ an acceleration wave if the first derivatives of $\mathbf{v}$ and $\mathbf{H}$ on Σ are not continuous. The jumps of $\mathbf{v}$ and $\mathbf{H}$ are:

$$[\![\dot{v}_i]\!] \neq 0, \qquad [\![H_{lk,j}]\!] \neq 0.$$

The unkown functions must satisfy at the surface Σ

$$(2.10)\quad \begin{aligned} &[\![\dot{v}_1]\!] - \frac{1}{\varrho}(c_{11}[\![H_{11,1}]\!]+b\lambda_2^2[\![H_{12,2}]\!]+b\lambda_1^2[\![H_{21,2}]\!]+c_{12}[\![H_{22,1}]\!]) = 0, \\ &[\![\dot{v}_2]\!] - \frac{1}{\varrho}(c_{21}[\![H_{11,2}]\!]+b\lambda_2^2[\![H_{12,1}]\!]+b\lambda_1^2[\![H_{21,1}]\!]+c_{22}[\![H_{22,2}]\!]) = 0, \\ &[\![\dot{H}_{i,j}]\!]-[\![v_{i,j}]\!] = 0. \end{aligned}$$

Introducing the amplitude $s_l \equiv n^in^j[\![H_{li,j}]\!]$ and using the compatability condition: $[\![f_{,t}]\!] = -\lambda n^k[\![f_{,k}]\!]$ (for the arbitrary function f whose first derivatives are discontinuous at Σ, λ is the normal speed of propagation and $\mathbf{n}$ is normal to the surface Σ) we obtain two algebraic equations for s_1 and s_2:

$$(2.11)\qquad \begin{aligned} &\varrho\lambda^2 s_1 - \big(c_{11}(n^1)^2 s_1 + b\lambda_2^2(n^2)^2 s_1 + b\lambda_1^2 n^1 n^2 s_2 + c_{12} n^1 n^2 s_2\big) = 0,\\ &\varrho\lambda^2 s_2 - \big(c_{21} n^1 n^2 s_1 + b\lambda_2^2 n^1 n^2 s_1 + b\lambda_1^2 (n^1)^2 s_2 + c_{22}(n^2)^2 s_2\big) = 0. \end{aligned}$$

Thus λ must be a root of

$$(2.12)\qquad \varrho^2\lambda^4 - \varrho\lambda^2\{(c_{11}+b\lambda_1^2)(n^1)^2 + (c_{22}+b\lambda_2^2)(n^2)^2\} + \{c_{11}(n^1)^2 + b\lambda_2^2(n^2)^2\}\{b\lambda_1^2(n^1)^2 + c_{22}(n^2)^2\} - (c_{21}+b\lambda_2^2)^2(n^1n^2)^2 = 0.$$

For the special case of a Hadamard hyperelastic material, for which $W(\mathrm{I}, \mathrm{II}, \mathrm{III}) = \varkappa\mathrm{I} + \gamma\mathrm{II} + F(\mathrm{III})$ (F is an arbitrary function of III), the coefficients c_{11}, c_{22} and c_{21} have the simple form:

$$(2.13)\qquad \begin{aligned} c_{11} &= b\lambda_1^2 + 2\mathrm{III}^{-1/2}\{\lambda_1^2\lambda_2^2\gamma + \mathrm{III}F'(\mathrm{III}) + 2\mathrm{III}^2F''(\mathrm{III})\} \equiv b\lambda_1^2 + \bar{A},\\ c_{22} &= b\lambda_2^2 + 2\mathrm{III}^{-1/2}\{\lambda_1^2\lambda_2^2\gamma + \mathrm{III}F'(\mathrm{II}) + 2\mathrm{III}^2F''(\mathrm{III})\} = b\lambda_2^2 + \bar{A},\\ c_{21} &= -b\lambda_2^2 + 2\mathrm{III}^{-1/2}\{\lambda_1^2\lambda_2^2\gamma + \mathrm{III}F'(\mathrm{III}) + 2\mathrm{III}^2F''(\mathrm{III})\} \equiv -b\lambda_2^2 + \bar{A},\\ b &\equiv 2\mathrm{III}^{-1/2}(\varkappa + \lambda_3^2\gamma), \qquad \bar{A} \equiv 2\mathrm{III}^{-1/2}\{\lambda_1^2\lambda_2^2\gamma + \mathrm{III}F' + 2\mathrm{III}^2F''\}. \end{aligned}$$

Using Eq. (2.13), Eq. (2.12) gives

$$(2.14)\qquad \varrho^2\lambda^4 - \varrho\lambda^2\{b(\lambda_1^2(n^1)^2 + \lambda_2^2(n^2)^2) + (c_{11}(n^1)^2 + c_{22}(n^2)^2)\} + b\big(\lambda_1^2(n^1)^2 + \lambda_2^2(n^2)^2\big)\big(c_{11}(c^1)^2 + c_{22}(n^2)^2\big) = 0.$$

It is evident that real acceleration (plane waves exist for all $\mathbf{n}$ if the constants b, c_{11} and c_{22} are positive.

CONCLUSION 1

The necessary and sufficient conditions for propagating the acceleration wave in the Hadamard material occupying the half space $X_2 \geqslant 0$ are $b > 0$ and $\bar{A} > 0$.

Directly from Eq. (2.13), we have $\bar{A} = c_{22} - \lambda_2^2 b$. Thus

$$(2.15)\qquad \bar{A} > 0 \Leftrightarrow c_{22} - \lambda_2^2 b > 0.$$

Further, if we demand to satisfy the boundary condition (no traction $\sigma_{22} = 0$) when the body is subjected to a pure homogeneous deformation (2.3), then λ_1, λ_2, λ_3 are no longer arbitrary but between them there is a relation:

$$(2.16)\qquad \mathrm{III}F'(\mathrm{III}) = -\lambda_2^2[\varkappa + (\lambda_1^2 + \lambda_2^2)\gamma].$$

Putting Eq. (2.16) into Eq. (2.13) we obtain

$$(2.17)\qquad \begin{aligned} c_{11} &= (\lambda_1^2 - \lambda_2^2)b + c_{22},\\ c_{21} &= -2\lambda_2^2 b + c_{22},\\ c_{22} &= 4\mathrm{III}^{3/2}F'(\mathrm{III}). \end{aligned}$$

Requiring $c_{11} > 0$ and $c_{22} > 0$ lead to

$$(2.18)\qquad \begin{aligned} (\lambda_1^2 - \lambda_2^2)(\varkappa + \lambda_3^2\gamma) + 2\mathrm{III}F'' &> 0,\\ F'' &> 0. \end{aligned}$$

Additionally, inequalities must be true which result directly from Eq. (2.16):

$$F' < 0,\qquad \gamma+\lambda_3^2F' < 0. \tag{2.19}$$

When the surface is free of traction so that Eq. (2.16) holds, the acceleration waves may propagate only when $\lambda_1, \lambda_2, \lambda_3$ satisfy the inequalities (2.18).

3. Rayleigh wave

In this section we discuss the existence and properties of Rayleigh waves for Hadamard materials. After HAYES, RIVLIN [2] we can look for a solution of the system (2.9) when the displacement components u_i are given by

$$\begin{aligned} u_1 &= f_1(x_2)\exp i(px_1+qt),\\ u_2 &= f_2(x_2)\exp i(px_1+qt),\\ u_3 &= 0, \end{aligned} \tag{3.1}$$

where $p > 0$ and $q < 0$.

The displacement (3.1) will represent the Rayleigh wave if u_1, u_2 are to tend to zero as x_2 tends to infinity. The general solution of type (3.1) (obtained by HAYES, RIVLIN [2]) is

$$\begin{aligned} u_1 &= (c_{12}+\lambda_1^2b)ip(m_1Ae^{-m_1x_2}+m_2Be^{-m_2x_2})\exp i(px_1+qt),\\ u_2 &= \{(\varrho q^2-c_{11}p^2+m_1^2\lambda_2^2b)Ae^{-m_1x_2}+(\varrho q^2-c_{11}p^2+m_2^2\lambda_2^2b)Be^{-m_2x_2}\}\exp i(px_1+qt), \end{aligned} \tag{3.2}$$

where m_1, m_2 must be positive roots of

$$\lambda_2^2bc_{22}m^4+\{c_{22}(\varrho q^2-c_{11}p^2)+\lambda_2^2b(\varrho q^2-\lambda_1^2bp^2)+(c_{21}+\lambda_2^2b)^2p^2\}m^2 + (\varrho q^2-c_{11}p^2)(\varrho q^2-\lambda_1^2bp^2) = 0. \tag{3.3}$$

This equation in the case of Hadamard material takes the form

$$\lambda_2^2bc_{22}\left(\frac{m}{p}\right)^4-\{\lambda_2^2b[(\lambda_1^2-\tfrac{2}{2}\lambda)b+c_{22}-\beta]+c_{22}(\lambda_1^2b-\beta]\}\left(\frac{m}{p}\right)^2 + [(\lambda_1^2-\lambda_2^2)b+c_{22}-\beta][\lambda_1^2b-\beta] = 0, \tag{3.4}$$

where β is always positive through the definition

$$\beta \equiv \varrho q^2/p^2. \tag{3.5}$$

Since Eq. (3.4) must give positive values for m_1^2 and m_2^2, it follows that

$$(\lambda_1^2-\lambda_2^2)b+c_{22}-\beta > 0,\qquad \lambda_1^2b-\beta > 0. \tag{3.6}$$

The inequality (3.6) together with Eq. (2.15) give

CONCLUSION 2

For a Hadamard material occupying a half-space which is homogeneously deformed with the surface $X_2 = 0$ being free of traction, if the conditions of existence of acceleration waves are satisfied (Conclusion 1), then the roots of Eq. (3.4) are real if $\lambda_1^2b-\beta > 0$.

To determine an equation for β we shall use the boundary conditions, that there is no surface traction in the state of pure homogeneous deformation and in the state of superimposed infinitesimal deformation:

$$\sigma_{22} = 0, \quad \sigma_{11} u_{2,1} - \bar{\sigma}_{22} = 0, \quad \bar{\sigma}_{22} = 0, \quad \bar{\sigma}_{32} = 0. \tag{3.7}$$

Introducing the solution (3.2) into these conditions and in order that A and B be nonzero [2], we obtain the equation for β:

$$[(\lambda_1^2 - \lambda_2^2)b + c_{22} - \beta][(\lambda_1^2 - \lambda_2^2)b - \beta] = \{c_{22}[(\lambda_1^2 - \lambda_2^2)b - \beta] + 4\lambda_2^2 b[c_{22} - \lambda_2^2 b)]\} \frac{m_1}{p} \frac{m_2}{p}. \tag{3.8}$$

From Eq. (3.4) we find

$$\frac{m_1^2}{p^2} \frac{m_2^2}{p^2} = \frac{[(\lambda_1^2 - \lambda_2^2)b + c_{22}][\lambda_1^2 b - \beta]}{\lambda_2^2 b c_{22}}.$$

Then Eq. (3.8) can be written in the form

$$cz^3 + 5dcz^2 + z(11c - 16d)d^2 - cd^3 = 0, \tag{3.9}$$

where the following notations were introduced:

$$\begin{aligned} z &\equiv \lambda_1^2 b - \beta, \\ d &\equiv \lambda_2^2 b, \\ c &\equiv c_{22}. \end{aligned} \tag{3.10}$$

Of course z (cf. Conclusion 2) must be positive.

The polynomial $w(z) = cz^3 + 5dcz^2 + z(11c - 16d)d^3 - cd^3$ can have only one positive root z_0 (see Fig. 1).

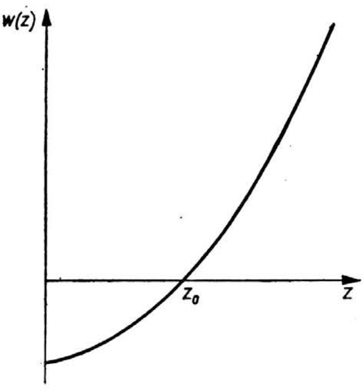

FIG. 1.

The root z_0 should be such that $\beta = \lambda_1^2 b - z_0 > 0$.

There are three possibilities:

$$\begin{aligned} \beta = 0 &\Leftrightarrow z_0 = \lambda_1^2 b \Leftrightarrow w(\lambda_1^2 b) = 0, \\ \beta > 0 &\Leftrightarrow z_0 < \lambda_1^2 b \Leftrightarrow w(\lambda_1^2 b) > 0, \\ \beta < 0 &\Leftrightarrow z_0 > \lambda_1^2 b \Leftrightarrow w(\lambda_1^2 b) < 0, \end{aligned} \tag{3.11}$$

where $w(z_0) = 0$ and

$$w(\lambda_1^2 b) = b^3\{c_{22}[\lambda_1^6 - \lambda_2^6 + 5\lambda_2^2\lambda_1^4 + 11\lambda_1^2\lambda_2^4] - 16\lambda_2^6\lambda_1^2 b\}, \tag{3.12}$$

when $w(\lambda_1^2 b) = 0$; it means that the initial static deformation is unstable (cf. USMANI, BEATTY [4]).

CONCLUSION 3

For Hadamard material the Rayleigh wave exists if $w(\lambda_1^2 b) > 0$. There is only one such wave and it is a retrograte.

Proof. To show that the Rayleigh wave given by Eq. (3.2) is retrograde, we must prove that

$$\operatorname{sign}\left(\frac{d\theta}{dt}\right)\Big|_{x_2=0} = -1,$$

where the angle θ is such that

$$\operatorname{tg}\theta = \frac{u_2^+}{u_1^+}$$

and u_1^+ and u_2^+ mean the real part of Eq. (3.2).

Using $\operatorname{sign}\left(\frac{d}{dt}\operatorname{tg}\theta\right) = \operatorname{sign}\left(\frac{d\theta}{dt}\right)$ and the boundary conditions (3.7), we have

$$\operatorname{sign}\left(\frac{d\theta}{dt}\right)\Big|_{x_2=0} = \operatorname{sign}\left[\frac{q}{p}\,\frac{(m_1+m_2)(\varrho q^2 - c_{11}p^2)}{(\varrho q^2 - c_{11}p^2) + m_1 m_2 c_{21}}\right] \tag{3.14}$$

$$= -\operatorname{sign}\left[\frac{(m_1+m_2)\left(\beta - (\lambda_1^2 - \lambda_2^2)b - c_{22}\right)}{\left(\beta - (\lambda_1^2 - \lambda_2^2)b - c_{22}\right) + \dfrac{m_1 m_2}{p_2}c_{21}}\right].$$

1. If $c_{21} \leqslant 0$, then Eq. (3.13) is true because of the inequalities (3.6).
2. If $c_{21} > 0$ which is equivalent to $c_{22} - 2\lambda_2^2 b > 0$, then using $c_{21} - c_{22} = -2\lambda_2^2 b$ and Eq. (3.6) we have that

$$\left(\beta - (\lambda_1^2 - \lambda_2^2)b - c_{22}\right) + \frac{m_1 m_2}{p^2}c_{21}$$

$$= \frac{[(\lambda_1^2 - \lambda_2^2)b + c_{22} - \beta]}{c_{22}[(\lambda_1^2 - \lambda_2^2)b - \beta] - 4\lambda_2^2 b[c_{22} - \lambda_2^2 b]}\{(c_{21} - c_{22})[(\lambda_1^2 - \lambda_2^2)b - \beta] - 4\lambda_2^2 b[c_{22} - \lambda_2^2 b]\}$$

is always negative and also in that case Eq. (3.13) is proved.

4. Examples of Hadamard material

In this section we discuss the condition $w(\lambda_1^2 b) > 0$ for two different models of Hadamard material

$$W = \varkappa \mathrm{I} + \gamma \mathrm{II} + F(\mathrm{III}).$$

Model 1: $F(\mathrm{III}) = -(\varkappa + 2\gamma)\ln \mathrm{III}$.

Model 2: $F(\mathrm{III}) = \frac{\varkappa+2\gamma}{\mathrm{III}}$.[1]

For the first model we calculate that

(4.1) $$w(\lambda_1^2 b) = 4\mathrm{III}^{-1/2} b^3 \{(\varkappa+2\gamma)(\lambda_1^6-\lambda_2^6+5\lambda_1^4\lambda_2^2+11\lambda_1^2\lambda_2^4)-8\lambda_1^2\lambda_2^6(\varkappa+\lambda_3^2\gamma)\}.$$

a. If $\lambda_1 \geqslant \lambda_2$, then we have always $w(\lambda_1^2 b) > 0$ which means that in this case a Rayleigh wave can propagate through the medium.

b. If $\lambda_1 < \lambda_2$, let us say $\lambda_2 = k\lambda_1$ and $k > 1$, then

(4.2) $$w(\lambda_1^2 b) = 4\mathrm{III}^{-1/2} b^3 \lambda_1^6 \{(1-k^6+5k^2+11k^4)(\varkappa+2\gamma)-8k^6\lambda_1^2(\varkappa+\lambda_3^2\gamma)\}.$$

Using the boundary condition $\sigma_{22} = 0$ (2.16):

(4.3) $$k^2\lambda_1^2[\varkappa+(\lambda_1^2+\lambda_3^2)\gamma] = \varkappa+2\gamma,$$

we have from Eq. (4.2)

(4.4) $$w(\lambda_1^2 b) = 4\mathrm{III}^{-1/2} b^3 \lambda_1^6 \{-k^6(\varkappa+2\gamma-8\lambda_1^4\gamma)+3k^4(\varkappa+2\gamma)+5k^2(\varkappa+2\gamma)+(\varkappa+2\gamma)\} \equiv 4\mathrm{III}^{-1/2} b^3 \lambda_1^6 N(k).$$

CONCLUSION 4

i) For the first model, for all $\lambda_1 = \lambda_1^0$ satisfying the inequality $\varkappa+2\gamma > 8(\lambda_1^0)^4\gamma$, there exists only one critical value k_0 such that $N(k_0) = 0$. Then for all $k > k_0$ or, equivalently, for all $\lambda_2 > \lambda_2^0$ where $\lambda_2^0 = k_0\lambda_1^0$, there is no Rayleigh wave since $N(k) < 0$ for $k > k_0$ (cf. Eq. $(3.1)_3$). We shall calculate from Eq. (4.3) the value λ_3^0 corresponding to the value $\lambda_1^0, \lambda_2^0, k_0$.

ii) If $\lambda_1 = \lambda_1^0$ is such that $\varkappa+2\gamma < 8(\lambda_1^0)^4\gamma$, then $N(k) > 0$ for all k. There is no critical value of k and the Rayleigh wave can propagate through the medium.

Similarly for the second model, for all $\lambda_1 \geqslant \lambda_2$ we have always $w(\lambda_1^2 b) > 0$. If $\lambda_2 = k\lambda_1$, $k > 1$ then

(4.5) $$w(\lambda_1^2 b) = 8b^3\lambda_1^6 \mathrm{III}^{-3/2}\{(\varkappa+2\gamma)(1-k^6+5k^2+11k^4)-4k^8\lambda_1^6\lambda_3^2(\varkappa+\lambda_3^2\gamma)\} = 8b^3\lambda_1^6\mathrm{III}^{-3/2}M(k).$$

$\sigma_{22} = 0$ means that

(4.6) $$k^4\lambda_1^6\lambda_3^2[\varkappa+(\lambda_1^2+\lambda_3^2)\gamma] = \varkappa+2\gamma.$$

The critical value of k is the solution of $M(k) = 0$ and Eq. (4.6); choosing k and λ_1 as known values we calculate λ_3^2 from Eq. (4.6) and put it into $M(k) = 0$.

Thus for the critical value k_0 we have the equation

(4.7) $$\overline{M}(\bar{z}) = \bar{z}^7 a_7+\bar{z}^6 a_6+\bar{z}^5 a_5+\bar{z}^4 a_4+\bar{z}^3 a_3+\bar{z}^2 a_2+\bar{z} a_1+a_0 = 0,$$

where $\bar{z} = k^2$ and $M(\bar{z}) = \overline{M}(k)$,

$$a_7 \equiv 4\lambda_1^8(\varkappa+\lambda_1^2\gamma), \quad a_6 \equiv (\varkappa+2\gamma)-4\lambda_1^8(\varkappa+\lambda_1^2\gamma),$$

$$a_5 \equiv -14(\varkappa+2\gamma)-20\lambda_1^8(\varkappa+\lambda_1^8\gamma), \quad a_4 \equiv 39(\varkappa+\lambda\gamma)-4\lambda_1^8(\varkappa+\lambda_1^2\gamma),$$

$$a_3 \equiv 68(\varkappa+2\gamma), \quad a_2 \equiv 39(\varkappa+2\gamma), \quad a_1 \equiv 10(\varkappa+2\gamma), \quad a_0 \equiv \varkappa+2\gamma.$$

[1] This function satisfies $\sigma_{\alpha\alpha}|_{\lambda_1=\lambda_2=\lambda_3=1} = 0$.

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