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BRIEF NOTES

The effect of a circular cylindrical hole on the stress intensity at a crack tip

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THE PAPER provides an analytical solution for the effect of a circular cylindrical hole (and rigid inclusion) on the stress intensity at a crack tip in an elastic material deforming under Mode III loading conditions. The results are compared with numerical solutions for the corresponding Mode I problems, and are used in a discussion of the effect of microcracks on the fracture toughness of brittle materials.

1. Introduction

IF A BRITTLE material, for example a rock-like material, contains a macroscopic crack, the onset of crack extension is not accompanied by plastic deformation, but instead microcracks form in a zone ahead of the crack tip [1, 2]. The microcrack size is typically that of the structural element (e.g. the crystal size), and the microcracks form because there is a distribution of failure stresses for these elements. It has been surmised [1] that the microcracks around a primary crack may be regarded as being contained within two zones: (a) an inner zone very close to the crack tip where the microcracks interact or link with the primary crack and so provide the principal driving force for macroscopic crack extension, and (b) an outer zone in front of the crack tip where the microcracks reduce the effective modulus of the material within that zone.

EVANS, HEUER and PORTER [1] argue that this reduction in the modulus within the outer microcrack zone is a source of toughness enhancement, i.e. is responsible for an increase in K_{IC}, the magnitude of the crack tip stress intensity at the onset of crack extension. To support their arguments, they rely on theoretical results [3, 4] for a crack whose tip lies within an elastic inclusion of lower modulus. These results in dicatethat, for a given applied stress and crack length, the crack opening decreases as the inclusion modulus decreases. Thus Evans, Heuer and Porter argue that the crack opening decreases as the density of microcracks in the outer zone increases, and also as the size of this zone increases, thereby causing a corresponding increase in the material's resistance to the onset of crack extension. However, the present author believes that it is inappropriate, when considering the effects of this outer zone, to use the results for a model in which a crack tip lies within an elastic inclusion of lower modulus. It is more appropriate to use the results from a model in which an elastic inclusion of lower modulus lies ahead of the crack tip, i.e. the crack does not penetrate the inclusion. In this context TIROSH and TETELMAN [5] have analysed the model of a solid containing a circular cylindrical hole ahead of a crack tip, the hole centre lying on the crack plane and the solid deforming under Mode I plane strain conditions. Their numerical results show that the hole produces an increase, not a decrease, in the crack tip stress intensity, the magnitude of this increase being greater the larger is the hole radius, and the nearer is the hole centre to the crack tip. On this reckoning, the author takes the alternative view to that of Evans, Heuer and Porter, and believes that the outer microcrack zone's presence is a source of weakness rather than toughness enhancement.

This paper addresses the corresponding Mode III problem, for which it is possible to obtain analytical solutions. Thus the paper assesses the effect of a circular cylindrical hole on the stress intensity at a crack tip in an elastic material deforming under Mode III loading conditions. The results clearly show that the hole produces an increase in the crack tip stress intensity, the magnitude of this increase being greater the larger is the hole radius, and the nearer is the hole centre to the crack tip; there is therefore accord with the Mode I numerical results of TIROSH and TETELMAN [5]. Solutions are also obtained for the case where the hole is replaced by a rigid inclusion; in this case the rigid inclusion's presence produces a decrease in the crack tip stress intensity.

2. Theoretical analysis

The model is illustrated in Fig. 1. A semi-infinite crack exists within an infinite solid deforming under Mode III loading conditions, and there is a circular cylindrical hole of radius a situated ahead of the crack tip, the hole centre being at a distance s from the crack tip; the stress intensity due to the applied loadings is K_A in the hole's absence. With such a mode III problem, the displacement w, which is parallel to the figure normal at all points of the solid, satisfies Laplace's equation

$$\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}=0 \tag{2.1}$$

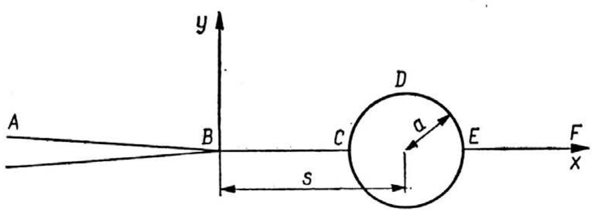

FIG. 1. The $z = x+iy$ plane containing a semi-infinite crack and a hole of radius a situated in front of the crack tip. The stress intensity due to the applied loadings is K_A in the hole's absence.

the appropriate stress components being

$$p_{xz}=\frac{\mu\partial w}{\partial x} \quad \text{and} \quad p_{yz}=\frac{\mu\partial w}{\partial y}, \tag{2.2}$$

where μ is the shear modulus of the material. Accordingly, there exists some complex function $F(z)$, where $z = x+iy$, such that the displacement is given by

$$\mu w = \mathrm{Re}[F(z)] \tag{2.3}$$

and the stresses by

$$\frac{dF}{dz} = p_{xz} - ip_{yz} = \frac{\mu \partial w}{\partial x} - \frac{i\mu \partial w}{\partial y}. \tag{2.4}$$

$F(z)$ has to satisfy the appropriate boundary conditions, and in order to determine its value, the z plane is mapped into some other $t(= \varepsilon + i\eta)$ plane where a solution can be readily obtained, using the relation

$$\frac{dF}{dt} = \frac{dF}{dz} \cdot \frac{dz}{dt} = \frac{\mu \partial w}{\partial \varepsilon} - \frac{i\mu \partial w}{\partial \eta}. \tag{2.5}$$

The conformal transformation that maps the region in the upper half of the z plane outside the cylindrical hole, into the upper half of the t plane, with the hole of diameter $2a$ transforming into a crack of length $2a$, and with corresponding points transforming as shown in Figs. 1 and 2, is

$$t - h = \frac{a}{2} \left[\frac{(z-s)}{a} + \frac{a}{(z-s)} \right], \tag{2.6}$$

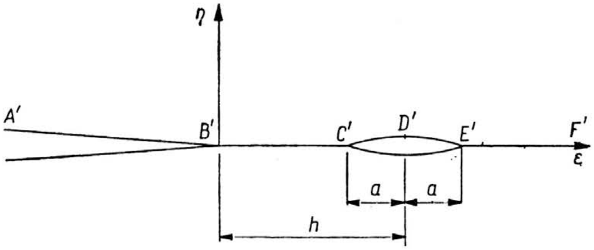

FIG. 2. The $t = \varepsilon + i\eta$ plane obtained from the z plane by the conformal transformation (2.6); corresponding points transform as shown (see also Fig. 1).

where h is the distance between the semi-infinite crack tip and the mid-point of the finite crack ahead of the semi-infinite crack in the t plane. The complex function that satisfies the boundary conditions in Fig. 2 is given by the relation

$$\frac{dF}{dt} = \frac{i(\pm A \pm Bt)}{\sqrt{t(t-h+a)(t-h-a)}}, \tag{2.7}$$

where A and B are real constants whose values remain to be determined (the $\pm$ signs are to ensure consistency with regard to the branch taken by the function). For large t, z, the relation (2.6) gives $t = z/2$, and since K_A is the stress intensity due to the applied loadings in the hole's absence, then

$$\frac{dF}{dz} = \frac{-iK_A}{\sqrt{2\pi z}} = \frac{\pm iB}{\sqrt{2z}} \tag{2.8}$$

using the relation (2.7), whereupon

$$K_A = \pm \sqrt{\pi} B. \tag{2.9}$$

For small t, and z, the relation (2.6) gives, in view of

$$h = \frac{(s^2+a^2)}{2s}, \tag{2.10}$$

the result

$$t = \frac{z(s^2-a^2)}{2s^2}. \tag{2.11}$$

Thus if K_L is the stress intensity at the tip of the semi-infinite crack in the original model, then

$$\frac{dF}{dz} = \frac{-iK_L}{\sqrt{2\pi z}} = \frac{\pm iA}{\sqrt{z}} \sqrt{\frac{s^2-a^2}{2s^2(h^2-a^2)}} \tag{2.12}$$

using the relations (2.7), (2.10) and (2.11), whereupon

$$K = \pm\sqrt{2\pi}A \sqrt{\frac{s^2-a^2}{2s^2(h^2-a^2)}}. \tag{2.13}$$

To ensure that the total dislocation content within the cylindrical hole in the z plane, or within the crack in the t plane, is zero, the integral of $dw/d\varepsilon$ between the limits $\varepsilon = h-a$ and $\varepsilon = h+a$ must be zero, whereupon the relation (2.7) gives

$$\int_{-a}^{+a} \frac{[\pm A \pm B(\chi+h)]d\chi}{\sqrt{(\chi+h)(\chi^2-a^2)}} = 0 \tag{2.14}$$

an expression which simplifies to

$$\frac{A}{B} = \pm(a+h)\frac{\mathbf{E}(k)}{\mathbf{K}(k)}, \tag{2.15}$$

where $\mathbf{E}$ and $\mathbf{K}$ are complete elliptic integrals with $k^2 = 2m/(1+m)$ where $m = a/h$.

It finally follows from the relations (2.9), (2.10), (2.13) and (2.15) that the ratio of the stress intensity at the crack tip in the original model to the stress intensity in the hole's absence is

$$\frac{K_L}{K_A} = \sqrt{1-\lambda^2}\,\frac{(1-\lambda)}{(1+\lambda)}\,\frac{\mathbf{E}(k)}{\mathbf{K}(k)} \tag{2.16}$$

with

$$k^2 = 1-\left(\frac{1-\lambda}{1+\lambda}\right)^2 \tag{2.17}$$

and with $\lambda = a/s$. The relations (2.16) and (2.17) enable the ratio K_L/K_K to be determined as a function of the parameter $\lambda = a/s$, and the results are shown in Table 1. These results clearly show that the hole produces an increase in the crack tip stress intensity, the magnitude of the increase being greater the larger is the hole radius, and the nearer is the hole centre to the crack tip; there is therefore accord with the Mode I numerical results of TIROSH and TETELMAN [5].

The preceding analysis is easily extended to the case where the cylindrical hole is replaced by a rigid inclusion, full cohesion being maintained between the inclusion and the surrounding material. The same conformal transformation is used, but the crack of length $2a$ in the t plane for the hole problem disappears, and the required complex function satisfies the relation

$$\frac{dF}{dt} = \frac{-iA}{\sqrt{t}}. \tag{2.18}$$

Table 1

$\lambda = \frac{a}{s}$	$\frac{K_L}{K_A}$ (hole)	$\frac{K_L}{K_A}$ (inclusion)
0	1.000	1.000
0.1	1.005	0.995
0.2	1.016	0.980
0.3	1.048	0.954
0.4	1.092	0.917
0.5	1.142	0.866
0.6	1.214	0.800
0.7	1.333	0.714
0.8	1.485	0.600
0.9	3.005	0.436

The ratio of the stress intensity (K_L) at the crack tip to the stress intensity (K_A) in the hole's absence, as a function of the parameter $\lambda = a/s$; a is the hole radius and s is the distance between the crack tip and the hole centre. The results are also given for a rigid inclusion.

For large t, z, it follows, by use of arguments similar to those adopted for the circular cylindrical hole model, that

$$K_A = +\sqrt{\pi}A, \tag{2.19}$$

where K_A is the stress intensity due to the applied loadings in the inclusion's absence. Similarly, by considering the small t, z, case it follows that the stress intensity K_L at the tip of the semi-infinite crack in the original model is given by the expression

$$K_L = \sqrt{\frac{\pi(s^2-a^2)}{s^2}}. \tag{2.20}$$

It finally follows from the relations (2.19) and (2.20) that the ratio of the stress intensity at the crack tip in the original model to the stress intensity in the inclusion's absence is

$$\frac{K_L}{K_A} = \sqrt{1-\lambda^2} \tag{2.21}$$

with λ again equal to a/s. The results (Table 1) clearly show that, in contrast with a hole, the inclusion produces a decrease in the crack tip stress intensity, the magnitude of the decrease being greater the larger is the inclusion radius, and the nearer is the inclusion centre to the crack tip; again there is accord with the Mode I numerical results of TIROSH and TETELMAN [5].

3. Discussion

The preceding section's theoretical results clearly show that the presence of a cylindrical hole ahead of a crack in an elastic material leads to an increase in the crack tip stress intensity, while the presence of a rigid inclusion leads to a reduction in the stress intensity. It follows that if the circular cylindrical hole is replaced by a cylinder with a modulus that is less than the modulus of the surrounding material, then the stress intensity is also enhanced. The results, which complement the Mode I numerical results of TIROSH and TETELMAN [5], provide further support for the view, developed in the Introduction, that the microcracks that form ahead of a macroscopic crack in some brittle materials as a result of the structural elements having a distribution of failure stresses, are a source of weakness rather than toughness enhancement, i.e. they are responsible for a reduction in K_{IC} the magnitude of the crack tip stress intensity at the onset of crack extension.

However, since the structural elements have a distribution of failure stresses, the macroscopic crack front will be irregular, and crack extension will be more difficult, i.e. K_{IC} will increase as a result of this irregularity. It is in this respect that the author believes that microcracking is responsible for larger than expected K_{IC} values, rather than by a reduced modulus effect from the other microcrack zone.

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