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ON THE GROUP OF POINTS G_4^1 ON A SEXTIC CURVE WITH FIVE DOUBLE POINTS.

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The present note relates to a special group of points considered incidentally by MM. Brill and Nöther in their paper "Ueber die algebraischen Functionen und ihre Anwendung in der Geometrie," *Math. Annalen*, t. VII. pp. 268—310 (1874).

I recall some of the fundamental notions. We have a basis-curve which to fix the ideas may be taken to be of the order n, $=p+1$, with $\frac{1}{2}p(p-3)$ dps, and therefore of the "Geschlecht" or deficiency p; any curve of the order $n-3$, $=p-2$ passing through the $\frac{1}{2}p(p-3)$ dps is said to be an adjoint curve. We may have, on the basis-curve, a special group G_Q^q of Q points ($Q \ngtr 2p-2$); viz. this is the case when the Q points are such that every adjoint curve through $Q-q$ of them—that is, every curve of the order $p-2$ through $\frac{1}{2}p(p-3)$ dps and the $Q-q$ points—passes through the remaining q points of the group: the number q may be termed the "speciality" of the group: if $q=0$, the group is an ordinary one.

It may be observed that a special group G_Q^q is chiefly noteworthy in the case where $Q-q$ is so small that the adjoint curve is not completely determined: thus if $p=5$, viz. if the basis-curve be a sextic with 5 dps, then we may have a special group G_6^2, but there is nothing remarkable in this; the 6 points are intersections with the sextic of an arbitrary cubic through the 5 dps—the cubic of course intersects the sextic in the 5 dps counting as 10 points, and in 8 other points—and such cubic is completely determined by means of the 5 dps and any 4 of the 6 points. But contrariwise, there is something remarkable in the group G_4^1 about to be considered: viz. we have here on the sextic 4 points, such that every cubic through the 5 dps and through 3 of the 4 points (through 8 points in all) passes through the remaining one of the 4 points.

The whole number of intersections of the basis-curve with an adjoint, exclusive of the dps counting as $p(p-3)$ points, is of course $=2p-2$: hence an adjoint through the Q points of a group G_Q^q meets the basis-curve besides in R, $=2p-2-Q$, points; we have then the "Riemann-Roch" theorem that these R points form a special group G_R^r, where

$$Q + R = 2p - 2,$$

as just mentioned, and

$$Q - R = 2q - 2r;$$

viz. dividing in any manner the $2p-2$ intersections of the basis-curve by an adjoint into groups of Q and R points respectively, these will be special groups, or at least one of them will be a special group, G_Q^q, G_R^r, such that their specialities q, r are connected by the foregoing relation $Q - R = 2q - 2r$.

The Authors give (*l.c.*, p. 293) a Table showing for a given basis-curve, or given value of p, and for a given value of r, the least value of R and the corresponding values of q, Q: this table is conveniently expressed in the following form.

The least value of

$$R = p - \frac{p}{r+1} + r;$$

and then

$$q = \frac{p}{r+1} - 1,$$

$$Q = p + \frac{p}{r+1} - r - 2,$$

where $\frac{p}{r+1}$ denotes the integer equal to or next less than the fraction.

It is, I think, worth while to present the table in the more developed form:

n	p	Dps	$r=$ 1	2	3	4	5	6
4	3	0	G_3^1	G_4^2	.	.	.	.
			G_1^0	G_0^0	.	.	.	.
5	4	2	G_3^1	G_5^2	G_6^3	.	.	.
			G_3^1	G_1^0	G_0^0	.	.	.
6	5	5	G_4^1	G_6^2	G_7^3	G_8^4	.	.
			G_4^1	G_2^0	G_1^0	G_0^0	.	.
7	6	9	G_4^1	G_6^2	G_8^3	G_9^4	G_{10}^5	.
			G_6^2	G_4^1	G_2^0	G_1^0	G_0^0	.
8	7	14	G_5^1	G_7^2	G_9^3	G_{10}^4	G_{11}^5	G_{12}^6
			G_7^2	G_5^1	G_3^0	G_2^0	G_1^0	G_0^0
⋮								

where the table shows the values of $\dfrac{G_R^r}{G_Q^q}$ for any given values of p, r.

I recur to the case $p = 5$ and the group G_4^1, which is the subject of the present note: viz. we have here a sextic curve with 5 dps, and on it a group of 4 points G_4^1, such that every cubic through the 5 dps and through 3 points of the group, 8 points in all, passes through the remaining 1 point.

MM. Brill and Nöther show (by consideration of a rational transformation of the whole figure) that, given 2 points of the group, it is possible, and possible in 5 different ways, to determine the remaining 2 points of the group.

I remark that the 5 dps and the 4 points of the group form "an ennead" or system of the nine intersections of two cubic curves: and that the question is, given the 5 dps and 2 points on the sextic, to show how to determine on the sextic a pair of points forming with the 7 points an ennead: and to show that the number of solutions is $= 5$.

We have the following "Geiser-Cotterill" theorem:

If seven of the points of an ennead are fixed, and the eighth point describes a curve of the order n passing $a_1, a_2, .., a_7$ times through the seven points respectively, then will the ninth point describe a curve of the order ν passing $\alpha_1, \alpha_2, .., \alpha_7$ times through the seven points respectively: where

$$\nu = 8n - 3\Sigma a,$$

$$\alpha_1 = 3n - a_1 - \Sigma a,$$

$$\vdots$$

$$\alpha_7 = 3n - a_7 - \Sigma a,$$

and conversely

$$n = 8\nu - 3\Sigma\alpha,$$

$$a_1 = 3\nu - \alpha_1 - \Sigma\alpha,$$

$$\vdots$$

$$a_7 = 3\nu - \alpha_7 - \Sigma\alpha.$$

(Geiser, *Crelle-Borchardt*, t. LXVII. (1867), pp. 78—90; the complete form, as just stated, and which was obtained by Mr Cotterill, has not I believe been published): and also Geiser's theorem "the locus of the coincident eighth and ninth points is a sextic passing twice through each of the seven points."

The sextic and the curve n intersect in $6n$ points, among which are included the seven points counting as $2\Sigma a$ points: the number of the remaining points is $= 6n - 2\Sigma a$. Similarly, the sextic and the curve ν intersect in 6ν points, among which are included the seven points counting as $2\Sigma\alpha$ points: the number of the remaining points is $6\nu - 2\Sigma\alpha$ $(= 6n - 2\Sigma a)$. The points in question are, it is clear, common intersections of the sextic, and the curves n, ν: viz. of the intersections of the curves n, ν, a number $6n - 2\Sigma a$, $= 6\nu - 2\Sigma\alpha$, $= 3n + 3\nu - \Sigma a - \Sigma\alpha$ lie on the sextic.

The curves n, ν intersect in $n\nu$ points, among which are included the seven points counting $\Sigma a\alpha$ times: the number of the remaining intersections is therefore

$n\nu - \Sigma a\alpha$, but among these are included the $3n + 3\nu - \Sigma a - \Sigma \alpha$ points on the sextic; omitting these, there remain $n\nu - 3(n+\nu) - \Sigma a\alpha + \Sigma a + \Sigma \alpha$ points, or, what is the same thing, $(n-3)(\nu-3) - \Sigma(a-1)(\alpha-1) - 2$ points: it is clear that these must form pairs such that, the eighth point being either point of a pair, the ninth point will be the remaining point of the pair: the number of pairs is of course

$$\tfrac{1}{2}[(n-3)(\nu-3) - \Sigma(a-1)(\alpha-1) - 2],$$

and we have thus the solution of the question, given the seven points to determine the number of pairs of points on the curve n (or on the curve ν) such that each pair may form with the seven points an ennead.

In particular, if $n = 6$; $a_1, a_2, a_3, a_4, a_5, a_6, a_7 = 2, 2, 2, 2, 2, 1, 1$ respectively, viz. if the curve be a sextic having 5 of the points for dps, and the remaining two for simple points, then we find $\nu = 12$; $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7 = 4, 4, 4, 4, 4, 5, 5$ respectively, and the number of pairs is

$$= \tfrac{1}{2}[3.9 - 5(2-1)(4-1) - 2], = \tfrac{1}{2}(27 - 15 - 2), = 5,$$

viz. starting with the 5 dps and any 2 points of the group G_4^1 we can, in 5 different ways, determine the remaining 2 points of the group.

In reference to the number $3p-3$ of parameters in the curves belonging to a given value of p, it may be remarked as follows. Such a curve is rationally transformable into a curve of the order $p+1$ with $\tfrac{1}{2}p(p-3)$ dps, and therefore containing $\tfrac{1}{2}(p+1)(p+4), -\tfrac{1}{2}p(p-3), = 4p+2$ parameters. Employing an arbitrary homographic transformation to establish any assumed relations between the parameters, the number is diminished to $4p+2-8, = 4p-6$; and again employing a rational transformation by means of adjoint curves of the order $p-2$ drawn through the dps and $p-3$ points of the curve—thereby transforming the curve into one of the same order $p+1$ and deficiency p—then, assuming that the $p-3$ parameters (or constants on which depend the positions of the $p-3$ points) can be disposed of so as to establish $p-3$ relations between the parameters and so further diminish the number by $p-3$, the required number of parameters will finally be $4p-6-(p-3) = 3p-3$.

Cambridge, 26th October, 1874.