# 615.

## ON THE CONIC TORUS.

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THE equation $p+\surd(qr)+\surd(st)=0$, where $p, q, r, s, t$ are linear functions of the coordinates $(x, y, z, w)$, and as such are connected by a linear relation, belongs to a quartic surface having a nodal conic ($p=0$, $qr-st=0$); and four nodes (conical points), viz. these are the intersections of the line $q=0$, $r=0$ with the quadric surface $p^2-qr-st=0$, and of the line $r=0$, $s=0$ with the same surface. The quartic surface has also four tropes (planes which touch the surface along a conic); viz. these are the planes $q=0$, $r=0$, $s=0$, $t=0$, the conic of contact or tropal conic in each plane being the intersection of the plane with the before-mentioned quadric surface $p^2-qr-st=0$. The planes $q=0$, $r=0$, and also the conics in these planes pass through two of the nodes, say $A$, $C$; and the planes $s=0$, $t=0$, and also the conics in these planes pass through the remaining two nodes, say $B$, $D$; so that the relations of the surface are as is shown in fig. 1. It is to be added that $AB$, $BC$, $CD$, $DA$ (but not $AC$ or $BD$) are lines on the surface.

The planes $q=0$, $r=0$, which contain the tropal conics through $A$, $C$, are in general distinct from the planes $ABC$, $ADC$ which contain the line-pairs $BA$, $BC$ and $DA$, $DC$ respectively: and so also the planes $s=0$, $t=0$, which contain the tropal conics through $B$, $D$, are in general distinct from the planes $ABD$, $CBD$ which contain the line-pairs $AB$, $AD$ and $CB$, $CD$ respectively.

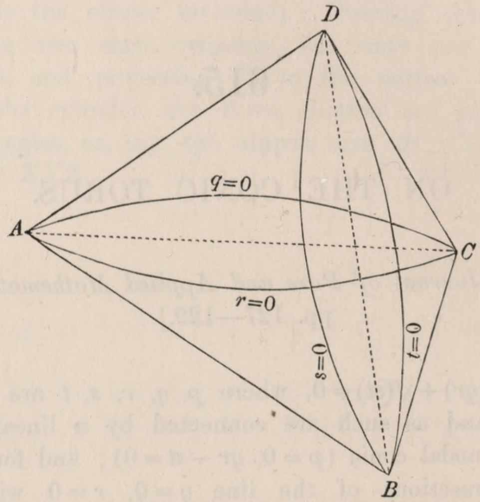
If, however, the identical linear relation contain only $p$, $s$, $t$, then the planes $q=0$, $r=0$ will be the planes $ABC$, $ADC$ respectively: and the tropal conics in these planes will consequently be the line-pairs $BA$, $BC$, and $DA$, $DC$ respectively. But the planes $s=0$, $t=0$ will continue to be distinct from the planes $ABD$, $CBD$: and the tropal conics in the planes $s=0$, $t=0$ will remain proper conics.

A surface of the last-mentioned form is

$$mz + \sqrt{(xy)} + \sqrt{(w^2 - z^2)} = 0,$$

viz. this has the nodal conic $z = 0$, $xy - w^2 = 0$, the nodes $\{x = 0, y = 0, (m^2 + 1) z^2 - w^2 = 0\}$, and $(z = 0, w = 0, x = 0)$, $(z = 0, w = 0, y = 0)$, and the tropes $x = 0$, $y = 0$, $z + w = 0$, $z - w = 0$; but the planes $z + w = 0$ and $z - w = 0$ are ordinary tropal planes each touching the surface in a proper conic; the planes $x = 0$, $y = 0$ special planes each touching along a line-pair.

Fig. 1.




The equation in question, writing therein $w = 1$ and $x + iy$, $x - iy$ in place of $(x, y)$ respectively, is

$$\{\sqrt{(x^2 + y^2)} + mz\}^2 = 1 - z^2,$$

which is derived from

$$(x + mz)^2 = 1 - z^2,$$

by the change of $x$ into $\sqrt{(x^2 + y^2)}$; and the surface is consequently the torus generated by the rotation of the conic $(x + mz)^2 = 1 - z^2$ about its diameter. Or, what is the same thing, the surface

$$mz + \sqrt{(xy)} + \sqrt{(w^2 - z^2)} = 0,$$

regarding therein $(x, y)$ as circular coordinates and $w$ as being $= 1$, is a torus. The rational equation is $U = 0$, where we have

$$\begin{aligned} U &= \{(m^2 + 1) z^2 - w^2 + xy\}^2 - 4m^2 z^2 xy \\ &= \{xy + (1 - m^2) z^2 - w^2\}^2 + 4m^2 z^2 (z^2 - w^2) \\ &= x^2y^2 + (m^2 + 1)^2 z^4 + w^4 + (2 - 2m^2) z^2 xy - (2 + 2m^2) z^2 w^2 - 2xyw^2. \end{aligned}$$

I find that the Hessian $H$ of this function $U$ contains the factor $xy + (1 - m^2) z^2 - w^2$, viz. that we have

$$H = \{xy + (1 - m^2) z^2 - w^2\} H',$$

where

$$\begin{aligned} H' = \quad & x^3y^3\ (1-m^2) \\ & + x^2y^2\{(3+8m^2+m^4)\,z^2 + (-3+m^2)\,w^2\} \\ & + xy\ \ \{(3+11m^2+9m^4+m^6)\,z^4 + (-6-12m^2+6m^4)\,z^2w^2 + (3+m^2)\,w^4\} \\ & + \quad (1+m^2)\{(1-m^2)\,z^2 - w^2\}\{(1+m^2)\,z^2 - w^2\}^2, \end{aligned}$$

giving without much difficulty

$$\begin{aligned} H' = \quad & z^6\ (1+m^2)^3(1-m^2) \\ & + 2z^4[(1+4m^2+m^4)\,xy - (1-m^4)\,w^2]\,(1+m^2) \\ & + \ z^2\ (xy-w^2)\,[(1+12m^2-m^4)\,xy - (1-m^4)\,w^2] \\ & + \quad [(1-m^2)\,xy - (1+m^2)\,w^2]\ U; \end{aligned}$$

say this is

$$= z^2H'' + [(1-m^2)\,xy - (1+m^2)\,w^2]\ U,$$

where

$$\begin{aligned} H'' = \quad & z^4(1+m^2)^3\ (1-m^2) \\ & + 2z^2(1+m^2)\ [(1+4m^2+m^4)\,xy - (1-m^4)\,w^2] \\ & + \quad (xy-w^2)\,[(1+12m^2-m^4)\,xy - (1-m^4)\,w^2], \end{aligned}$$

or, what is the same thing,

$$\begin{aligned} H'' = \quad & x^2y^2(1+12m^2-m^4) \\ & + 2xy\,[(1+4m^2+m^4)(1+m^2)\,z^2 + (-1+6m^2)\,w^2] \\ & + \quad (1-m^4)\{(1+m^2)\,z^2 - w^2\}^2. \end{aligned}$$

It consequently appears that the complete spinode curve or intersection of the quartic surface and its Hessian, being of order $4\times 8$, $=32$, breaks up into

$$U=0,\ xy + (1-m^2)\,z^2 - w^2 = 0,$$

that is,

| | | |
|---|---|---|
| conic $z=0,\ xy-w^2=0$ twice, | order | 4 |
| conic $z+w=0,\ xy-m^2w^2=0,$ | „ | 2 |
| conic $z-w=0,\ xy-m^2w^2=0,$ | „ | 2 |

and

$$U=0,\ z^2H''=0,$$

that is,

| | | |
|---|---|---|
| $U=0,\ z^2=0,$ conic $z=0,\ xy-w^2=0$ four times, | „ | 8 |
| proper spinode curve $U=0,\ H''=0,$ | „ | 16 |
| | | 32; |

viz. the intersection is made up of the conic $z=0$, $xy-w^2=0$ six times, the conics $z\pm w=0$, $xy-m^2w^2=0$ each twice, and the proper spinode curve of the order 16.