625.

ON THE CONDITION FOR THE EXISTENCE OF A SURFACE CUTTING AT RIGHT ANGLES A GIVEN SET OF LINES.

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In a congruency or doubly infinite system of right lines, the direction-cosines α, β, γ of the line through any given point (x, y, z), are expressible as functions of x, y, z; and it was shown by Sir W. R. Hamilton in a very elegant manner that, in order to the existence of a surface (or, what is the same thing, a set of parallel surfaces) cutting the lines at right angles, $\alpha dx + \beta dy + \gamma dz$ must be an exact differential: when this is so, writing $V = \int (\alpha dx + \beta dy + \gamma dz)$, we have $V = c$, the equation of the system of parallel surfaces each cutting the given lines at right angles.

The proof is as follows:—If the surface exists, its differential equation is $\alpha dx + \beta dy + \gamma dz = 0$, and this equation must therefore be integrable by a factor. Now the functions α, β, γ are such that $\alpha^2 + \beta^2 + \gamma^2 = 1$, and they besides satisfy a system of partial differential equations which Hamilton deduces from the geometrical notion of a congruency; viz. passing from the point (x, y, z) to the consecutive point on the line, that is, to the point whose coordinates are $x + \rho\alpha$, $y + \rho\beta$, $z + \rho\gamma$ (ρ infinitesimal), the line belonging to this point is the original line; and consequently α, β, γ, considered as functions of x, y, z, must remain unaltered when these variables are changed into $x + \rho\alpha$, $y + \rho\beta$, $z + \rho\gamma$, respectively. We thus obtain the equations

$$\alpha \frac{d\alpha}{dx} + \beta \frac{d\alpha}{dy} + \gamma \frac{d\alpha}{dz} = 0,$$

$$\alpha \frac{d\beta}{dx} + \beta \frac{d\beta}{dy} + \gamma \frac{d\beta}{dz} = 0,$$

$$\alpha \frac{d\gamma}{dx} + \beta \frac{d\gamma}{dy} + \gamma \frac{d\gamma}{dz} = 0.$$

Combining herewith the equations obtained by differentiation of $\alpha^2+\beta^2+\gamma^2=1$, viz.

$$\alpha\frac{d\alpha}{dx}+\beta\frac{d\beta}{dx}+\gamma\frac{d\gamma}{dx}=0,$$

$$\alpha\frac{d\alpha}{dy}+\beta\frac{d\beta}{dy}+\gamma\frac{d\gamma}{dy}=0,$$

$$\alpha\frac{d\alpha}{dz}+\beta\frac{d\beta}{dz}+\gamma\frac{d\gamma}{dz}=0,$$

and subtracting the corresponding equations, we obtain three equations which may be written

$$\alpha : \beta : \gamma = \frac{d\beta}{dz}-\frac{d\gamma}{dy} : \frac{d\gamma}{dx}-\frac{d\alpha}{dz} : \frac{d\alpha}{dy}-\frac{d\beta}{dx},$$

or, what is the same thing,

$$\frac{d\beta}{dz}-\frac{d\gamma}{dy},\ \frac{d\gamma}{dx}-\frac{d\alpha}{dz},\ \frac{d\alpha}{dy}-\frac{d\beta}{dx}=k\alpha,\ k\beta,\ k\gamma,$$

and, multiplying by α, β, γ, and adding,

$$k=\alpha\left(\frac{d\beta}{dz}-\frac{d\gamma}{dy}\right)+\beta\left(\frac{d\gamma}{dx}-\frac{d\alpha}{dz}\right)+\gamma\left(\frac{d\alpha}{dy}-\frac{d\beta}{dx}\right).$$

We thus see that, if the function on the right-hand vanish, then $k=0$, and consequently also

$$\frac{d\beta}{dz}-\frac{d\gamma}{dy},\ \frac{d\gamma}{dx}-\frac{d\alpha}{dz},\ \frac{d\alpha}{dy}-\frac{d\beta}{dx}\ \text{each}=0;$$

viz. if the equation $\alpha dx+\beta dy+\gamma dz=0$ be integrable, then $\alpha dx+\beta dy+\gamma dz$ is an exact differential; which is the theorem in question.

But it is interesting to obtain the first mentioned set of differential equations from the analytical equations of a congruency, viz. these are $x=mz+p$, $y=nz+q$, where m, n, p, q are functions of two arbitrary parameters, or, what is the same thing, p, q are given functions of m, n; and therefore, from the three equations, m, n are given functions of x, y, z. And it is also interesting to express in terms of these quantities m, n, considered as functions of x, y, z, the condition for the existence of the set of surfaces.

We have

$$\alpha,\ \beta,\ \gamma=\frac{m}{R},\ \frac{n}{R},\ \frac{1}{R},\ \text{where } R=\sqrt{1+m^2+n^2};$$

and thence without difficulty

$$\left(\alpha\frac{d}{dx}+\beta\frac{d}{dy}+\gamma\frac{d}{dz}\right)\alpha=\frac{1}{R^4}\left[(1+n^2)\left(m\frac{dm}{dx}+n\frac{dm}{dy}+\frac{dm}{dz}\right)\ -mn\ \left(m\frac{dn}{dx}+n\frac{dn}{dy}+\frac{dn}{dz}\right)\right],$$

$$\left(\alpha\frac{d}{dx}+\beta\frac{d}{dy}+\gamma\frac{d}{dz}\right)\beta=\frac{1}{R^4}\left[\ -mn\ \left(\quad ,, \quad\right)+(1+m^2)\left(\quad ,, \quad\right)\right],$$

$$\left(\alpha\frac{d}{dx}+\beta\frac{d}{dy}+\gamma\frac{d}{dz}\right)\gamma=\frac{1}{R^4}\left[\ -\ \left(\quad ,, \quad\right)\ -\ \left(\quad ,, \quad\right)\right],$$

so that the required equations in α, β, γ will be satisfied if only

$$m\frac{dm}{dx}+n\frac{dm}{dy}+\frac{dm}{dz}=0,$$

$$m\frac{dn}{dx}+n\frac{dn}{dy}+\frac{dn}{dz}=0,$$

and it is to be shown that these equations hold good.

Writing for shortness $dp = Adm + Bdn$, $dq = Cdm + Ddn$, the equations of the line give

$$1 = z\frac{dm}{dx}+A\frac{dm}{dx}+B\frac{dn}{dx}, \qquad 0 = z\frac{dn}{dx}+C\frac{dm}{dx}+D\frac{dn}{dx},$$

$$0 = z\frac{dm}{dy}+A\frac{dm}{dy}+B\frac{dn}{dy}, \qquad 1 = z\frac{dn}{dy}+C\frac{dm}{dy}+D\frac{dn}{dy},$$

$$-m = z\frac{dm}{dz}+A\frac{dm}{dz}+B\frac{dn}{dz}, \qquad -n = z\frac{dn}{dz}+C\frac{dm}{dz}+D\frac{dn}{dz};$$

or, writing

$$\lambda,\ \mu,\ \nu = \frac{dm}{dy}\frac{dn}{dz}-\frac{dm}{dz}\frac{dn}{dy},\quad \frac{dm}{dz}\frac{dn}{dx}-\frac{dm}{dx}\frac{dn}{dz},\quad \frac{dm}{dx}\frac{dn}{dy}-\frac{dm}{dy}\frac{dn}{dx},$$

so that identically

$$\lambda\frac{dm}{dx}+\mu\frac{dm}{dy}+\nu\frac{dm}{dz}=0,$$

$$\lambda\frac{dn}{dx}+\mu\frac{dn}{dy}+\nu\frac{dn}{dz}=0,$$

then in each set, multiplying by λ, μ, ν and adding, so as to eliminate A, B, C, D, we find

$$\lambda - m\nu = 0, \quad \mu - n\nu = 0.$$

Substituting these values of λ, μ in the last preceding equations, ν divides out, and we have the two equations in question.

The foregoing equations give further

$$A,\ B,\ C,\ D = -z+\frac{1}{\nu}\frac{dn}{dy},\quad -\frac{1}{\nu}\frac{dm}{dy},\quad -\frac{1}{\nu}\frac{dn}{dx},\quad -z+\frac{1}{\nu}\frac{dm}{dx}.$$

Taking for α, β, γ the before-mentioned values, we find

$$\frac{d\alpha}{dy}-\frac{d\beta}{dx}=\frac{1}{R}\left(\frac{dm}{dy}-\frac{dn}{dx}\right)-\frac{m}{R^3}\left(m\frac{dm}{dy}+n\frac{dn}{dy}\right)-\frac{n}{R^3}\left(m\frac{dm}{dx}+n\frac{dn}{dx}\right)$$

$$=\frac{1}{R^3}\left\{(1+n^2)\frac{dm}{dy}-(1+m^2)\frac{dn}{dx}+mn\left(\frac{dm}{dx}-\frac{dn}{dy}\right)\right\};$$

and similarly, but using the equations

$$m\frac{dm}{dx}+n\frac{dm}{dy}+\frac{dm}{dz}=0, \quad m\frac{dn}{dx}+n\frac{dn}{dy}+\frac{dn}{dz}=0,$$

to eliminate the coefficients $\frac{dm}{dz}$, $\frac{dn}{dz}$ which in the first instance present themselves, we find

$$\frac{d\beta}{dz} - \frac{d\gamma}{dy} = \frac{m}{R^3}\left\{(1+n^2)\frac{dm}{dy} - (1+m^2)\frac{dn}{dx} + mn\left(\frac{dm}{dx} - \frac{dn}{dy}\right)\right\},$$

$$\frac{d\gamma}{dx} - \frac{d\alpha}{dz} = \frac{n}{R^3}\left\{\qquad ,, \qquad ,, \qquad ,, \qquad\right\}:$$

whence, multiplying by γ, α, β, and adding,

$$\alpha\left(\frac{d\beta}{dz} - \frac{d\gamma}{dy}\right) + \beta\left(\frac{d\gamma}{dx} - \frac{d\alpha}{dz}\right) + \gamma\left(\frac{d\alpha}{dy} - \frac{d\beta}{dx}\right)$$

$$= \frac{1}{1+m^2+n^2}\left\{(1+n^2)\frac{dm}{dy} - (1+m^2)\frac{dn}{dx} + mn\left(\frac{dm}{dx} - \frac{dn}{dy}\right)\right\};$$

or we have

$$(1+n^2)\frac{dm}{dy} - (1+m^2)\frac{dn}{dx} + mn\left(\frac{dm}{dx} - \frac{dn}{dy}\right) = 0$$

as the condition for the existence of the set of surfaces.

It is clear that the condition is satisfied when the lines are the normals of a given surface: seeking the surfaces which cut the lines at right angles, we obtain the parallel surfaces; and we are led to the theorem that any parallel surface is the locus of the extremity of a line of constant length measured off from each point of the surface along the normal—or, what is equivalent thereto, the parallel surface is the envelope of a sphere of constant radius having its centre on the surface. I will verify the theorem for the case of the ellipsoid. Taking X, Y, Z as the coordinates of a point on the ellipsoid $\frac{X^2}{a^2} + \frac{Y^2}{b^2} + \frac{Z^2}{c^2} = 1$, and x, y, z as current coordinates, the equations of the normal are

$$\frac{a^2}{X}(x - X) = \frac{b^2}{Y}(y - Y) = \frac{c^2}{Z}(z - Z), \ (=\lambda \text{ suppose}).$$

We have therefore

$$X,\ Y,\ Z = \frac{a^2 x}{a^2+\lambda},\ \frac{b^2 y}{b^2+\lambda},\ \frac{c^2 z}{c^2+\lambda},$$

and thence

$$\frac{a^2x^2}{(a^2+\lambda)^2} + \frac{b^2y^2}{(b^2+\lambda)^2} + \frac{c^2z^2}{(c^2+\lambda)^2} = 1,$$

an equation which determines λ as a function of x, y, z.

The direction-cosines α, β, γ of the normal are proportional to $\frac{X}{a^2}$, $\frac{Y}{b^2}$, $\frac{Z}{c^2}$, that is, to $\dfrac{x}{a^2+\lambda}$, $\dfrac{y}{b^2+\lambda}$, $\dfrac{z}{c^2+\lambda}$, and the equation $\alpha^2+\beta^2+\gamma^2=1$ then determines their absolute magnitudes: the equation $\alpha dx+\beta dy+\gamma dz=dV$ thus is

$$\frac{\dfrac{x\,dx}{a^2+\lambda}+\dfrac{y\,dy}{b^2+\lambda}+\dfrac{z\,dz}{c^2+\lambda}}{\sqrt{\dfrac{x^2}{(a^2+\lambda)^2}+\dfrac{y^2}{(b^2+\lambda)^2}+\dfrac{z^2}{(c^2+\lambda)^2}}}=dV,$$

viz. the left-hand side, considering therein λ as a given function of V, is an exact differential. We verify this by finding the value of V, viz. writing down the two equations

$$\frac{x^2}{(a^2+\lambda)^2}+\frac{y^2}{(b^2+\lambda)^2}+\frac{z^2}{(c^2+\lambda)^2}-\frac{V^2}{\lambda^2}=0,$$

$$\frac{x^2}{a^2+\lambda}+\frac{y^2}{b^2+\lambda}+\frac{z^2}{c^2+\lambda}-\frac{V^2}{\lambda}=1,$$

these are equivalent in virtue of the equation that determines λ; and it is to be shown that, regarding V as given by either of them, say by the second equation, we have for dV its foregoing value. In fact, differentiating the second equation, the term in $d\lambda$ disappears by virtue of the first equation, and the result is

$$\frac{x\,dx}{a^2+\lambda}+\frac{y\,dy}{b^2+\lambda}+\frac{z\,dz}{c^2+\lambda}-\frac{V\,dV}{\lambda}=0,$$

in which substituting for $\dfrac{V}{\lambda}$ its value from the first equation, we have for dV the value in question. Regarding V as a given constant, the two equations give, by elimination of λ, an equation $\phi(x, y, z, V)=0$, which is, in fact, the surface parallel to the ellipsoid and at a constant normal distance $=V$ from it.