31.

LECTURES ON THE PRINCIPLES OF UNIVERSAL ALGEBRA.

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LECTURE I.

PRELIMINARY CONCEPTIONS AND DEFINITIONS.

Apotheosis of Algebraical Quantity.

A MATRIX of a quadrate form historically takes its rise in the notion of a linear substitution performed upon a system of variables or carriers; regarded apart from the determinant which it may be and at one time was almost exclusively used to represent, it becomes an empty *schema* of operation, but in conformity with Hegel's principle that the Negative is the course through which thought arrives at another and a fuller positive, only for a moment loses the attribute of quantity to emerge again as quantity, if it be allowed that that term is properly applied to whatever is the subject of functional operation, of a higher and unthought of kind, and so to say, in a glorified shape,—as an organism composed of discrete parts, but having an essential and undivisible unity as a whole of its own. *Naturam expellas furcâ, tamen usque recurret**. The conception of multiple quantity thus rises upon the field of vision.

At first undifferentiated from their content, matrices came to be regarded as susceptible of being multiplied together; the word multiplication, strictly applicable at that stage of evolution to the content alone, getting transferred by a fortunate confusion of language to the schema, and superseding, to some extent, the use of the more appropriate word composition applied to the reiteration of substitution in the Theory of Numbers. Thus there came into view a process of multiplication which the mind, almost at a glance, is able to recognize must be subject to the associative law of ordinary

* *Chassez le naturel, il revient au galop*, a familiar quotation which I thought was from Boileau, but my friend Prof. Rabillon informs me is from a comedy of Destouches (born in 1680, died 1754).

multiplication, although not so to the commutative law; but the full significance of this fact lay hidden until the subject-matter of such operations had dropped its provisional mantle, its aspect as a mere schema, and stood revealed as *bona-fide* multiple quantity subject to all the affections and lending itself to all the operations of ordinary numerical quantity. This revolution was effected by a forcible injection into the subject of the concept of addition, that is, by choosing to regard matrices as susceptible of being added to one another; a notion, as it seems to me, quite foreign to the idea of substitution, the *nidus* in which that of multiple quantity was laid, hatched and reared. This step was, as far as I know, first made by Cayley in his Memoir on Matrices, in the *Phil. Trans.* 1858, wherein he may be said to have laid the foundation-stone of the science of multiple quantity. That memoir indeed (it seems to me) may with truth be affirmed to have ushered in the reign of Algebra the 2nd; just as Algebra the 1st, in its character, not as mere art or mystery, but as a science and philosophy, took its rise in Harriot's *Artis Analyticae Praxis,* published in 1631, ten years after his death, and exactly 250 years before I gave the first course of lectures ever delivered on Multinomial Quantity, in 1881, at the Johns Hopkins University. Much as I owe in the way of fruitful suggestion to Cayley's immortal memoir, the idea of subjecting matrices to the additive process and of their consequent amenability to the laws of functional operation was not taken from it, but occurred to me independently before I had seen the memoir or was acquainted with its contents; and indeed forced itself upon my attention as a means of giving simplicity and generality to my formula for the powers or roots of matrices, published in the *Comptes Rendus* of the Institute for 1882 (Vol. XCIV. pp. 55, 396). My memoir on Tchebycheff's method concerning the totality of prime numbers within certain limits, was the indirect cause of turning my attention to the subject, as (through the systems of difference-equations therein employed to contract Tchebycheff's limits) I was led to the discovery of the properties of the latent roots of matrices, and had made considerable progress in developing the theory of matrices considered as quantities, when on writing to Prof. Cayley upon the subject he referred me to the memoir in question: all this only proves how far the discovery of the quantitative nature of matrices is removed from being artificial or factitious, but, on the contrary, was bound to be evolved, in the fulness of time, as a necessary sequel to previously acquired cognitions.

Already in Quaternions (which, as will presently be seen, are but the simplest order of matrices viewed under a particular aspect) the example had been given of Algebra released from the yoke of the commutative principle of multiplication—an emancipation somewhat akin to Lobatchewsky's of Geometry from Euclid's noted empirical axiom; and later on, the Peirces, father and son (but subsequently to 1858) had prefigured the universalization of Hamilton's theory, and had emitted an opinion to the effect that probably all systems of algebraical symbols subject to the associative law of multiplication would be eventually found to be identical with linear transformations of schemata susceptible of matricular representation.

That such must be the case it would be rash to assert; but it is very difficult to conceive how the contrary can be true, or where to seek, outside of the concept of substitution, for matter affording pabulum to the principle of free consociation of successive actions or operations.

Multiplication of Matrices.

A matrix written in the usual form may be regarded as made up of parallels of latitude and of longitude, so that to every term in one matrix corresponds a term of the same latitude and longitude in any other of the same order.

Every matrix possesses a principal axis, namely, the diagonal drawn from the intersection of the first two parallels to the intersection of the last two of latitude and longitude; and by a symmetrical matrix is always to be understood one in which the principal diagonal is the axis of symmetry. If there were ever occasion to consider a symmetrical matrix in which this coincidence does not exist, it might be called improperly symmetrical. This designation might and probably ought to be extended to matrices symmetrical, not merely in regard to the second visible diagonal, but to all the $(\omega - 1)$ rational diagonals of a matrix of the order ω, a rational diagonal being understood to mean any line straight or broken, drawn through ω elements, of which no two have the same latitude or longitude.

The composition of substitutions directly leads to the following rule for the multiplication of matrices. If m, n, be matrices corresponding to substitutions in which m is the antecedent or passive, and n the consequent or active, their product may be denoted by mn (that is, m multiplied by n), and then any term in the product of the two matrices will be equal to its parallel of latitude taken in the antecedent or passive and multiplied by its parallel of longitude taken in the consequent or active matrix. Cauchy has taught us what is to be understood by the product of one rectangular array or matrix by another of the same length and breadth, and we have only to consider the case of rectangles degenerating each to a single line and column respectively, to understand what is meant by the product of the multiplication of the two parallels spoken of above. It may, however, be sometimes convenient to speak of the *disjunctive product* of two sets of the same number of elements, meaning by this the sum of the products of each element in the one by the corresponding element in the other. Thus $(\lambda l)\, mn$ denoting the term in mn of latitude λ and longitude l, we have the equation

$$(\lambda l)\, mn = \lambda m \times ln,$$

where, of course, λm means the λth parallel of latitude, and ln the lth parallel of longitude, in m and n respectively. This notation may be extended so as to express the value of any minor determinant of mn; such minor may obviously be denoted by

$$\begin{array}{l} \lambda_1 l_1, \quad \lambda_1 l_2, \ldots \lambda_1 l_i, \\ \lambda_2 l_1, \quad \lambda_2 l_2, \ldots \lambda_2 l_i, \\ \ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots \\ \lambda_i l_1, \quad \lambda_i l_2, \ldots \lambda_i l_i, \end{array}$$

and its value will be the product of the two rectangles (in Cauchy's sense) formed respectively by the $\lambda_1, \lambda_2, \ldots \lambda_i$ parallels of latitude in m, and the $l_1, l_2, \ldots l_i$ parallels of longitude in n.

Any other definition of multiplication of matrices, such as the rule for multiplying lines by lines, or columns by columns, sins against good method, as being incompatible with the law of consociation, and ought to be inexorably banished from the text-books of the future. It is almost unnecessary to add that by a pth power of a matrix m is to be understood the result of multiplying p m's together; and by the qth root of m, a matrix which multiplied by itself q times produces m: hence we can attach a clear idea to any positive integral or fractional power. The complete extension of the ordinary theory of surds to multinomial quantity will appear a little further on. But it is well at this point to draw attention to the fact that at all events, if M, M' are positive integer powers of the same matrix m, the factors M, M' are convertible, that is, $MM' = M'M$, this commutative law being an immediate consequence (too obvious to insist upon) of the associative law of multiplication.

On Zero and Nullity.

The absolute zero for matrices of any order is the matrix all of whose elements are zero. It possesses so far as regards multiplication (and as will presently be evident as regards addition also) the distinguishing property of the ordinary zero, namely, that when entering into composition with any other matrix, either actively or passively, the product of such composition is itself over again; so that it may be said to absorb into itself any foreign matrix (of its own order) with which it is combined. This is the highest degree of nullity which any matrix can possess, and (regarded as an integer) will be called ω, the order of the matrix. On the other hand, if the matrix has finite content, its nullity will be regarded as zero. Between these two limits the nullity may have any integer value; thus, if its content, that is, its determinant, vanishes without any other special relation existing between its elements, the nullity will be called 1; if all the first minors vanish, 2; and, in general and more precisely, if all the minors of order $\omega - i + 1$ vanish, but the minors of order $\omega - i$ do not all vanish, the nullity will be said to be i: as an example, if the elements are not all zero, but every minor of the second order vanishes, the nullity is $\omega - 1$.

In general, a substitution impressed on a set of variables may be reversed, and the problem of reversal is perfectly determinate; but when the matrix—the schema of the substitution—is affected with any degree of nullity, such reversal becomes indeterminate. Hence the use of the word indeterminate employed by Cayley to characterize matrices affected with any degree of nullity, in which he has been followed by Clifford, who goes a step further in distinguishing the several degrees of indeterminateness from one another.

On Addition and Monomial Multiplication of Matrices.

The sum of two matrices of like order is the matrix of which each element is the sum of the elements of the same latitude and longitude as its own in the component matrices; thus, as stated by anticipation in what precedes, the addition of a zero matrix to any matrix of like order leaves the latter entirely unchanged.

Addition of matrices obviously will be subject to the same two associative and commutative laws as the addition of monomial quantities. This seems to me a sufficient ground for declining to accept *associative* as the distinguishing name of the algebra of multinomial quantity; for the emphasis thereby laid on association would seem to imply the entire absence of the commutative principle from the theory, whereas, although not having a place in multinomial multiplication, it flourishes in full vigour in the not less important, and, so to say, collateral process of multinomial addition. If k is any positive integer, the addition of the same matrix taken k times obviously leads to a matrix of which each element is k times the corresponding element of the given one; and if p times one matrix is q times another, the elements of the first are obviously $\frac{q}{p}$ into the corresponding ones of the other: hence, if k is any positive monomial quantity, k times a given matrix, by a legitimate use of language, should and will be taken to mean the matrix obtained by multiplying each element in the given one by k. And as the negative of a given matrix ought to mean the matrix which added to the given one should produce the zero-matrix previously defined, the meaning of multiplying a matrix by k may be extended, with the certainty of leading to no contradiction, to the case of any commensurable value of k positive or negative, and consequently, by the usual and valid course of inference, to the case of k being any monomial symbol whatever, whether possessing arithmetical content or not.

On the Multinomial Unit and Scalar Matrix.

On subjecting a matrix of any order ω to a resolution similar to that by which one of the second order may be resolved into a scalar and a vector, it will be shown hereafter that the ω^2 components separate into a group of $\omega^2 - 1$ terms analogous to the vector and to a single term analogous to the scalar of a quaternion. This outstanding single term is of an invariable form, namely, its principal diagonal consists of elements having the same value, which may be called its parameter, and all the other elements are zeros.

A matrix of such form I shall call a scalar. When the parameter is unity it may be termed a multinomial unity and denoted by $\overset{\omega}{\Upsilon}$*, or in place of ω we may write ω dots over Υ, or for greater simplicity when desirable write simply Υ. Any scalar, by virtue of what precedes, is a mere monomial multiplier of some such Υ.

Let $k\Upsilon$ be any scalar of order ω. It will readily be seen, by applying the laws of multiplication and addition previously laid down, that

$$\phi(k\Upsilon) = \phi(k) . \Upsilon, \text{ and that } k\Upsilon . m = m . k\Upsilon = km.$$

Thus a scalar possesses all the essential properties of a monomial quantity, and a multinomial unity of ordinary unity; in particular, the faculty of being absorbed in any other coordinate matrix with which it comes in contact. A scalar whose parameter vanishes of course becomes a zero-matrix.

The properties stated of a scalar $k\Upsilon$ serve to show that in all operations into which it enters the Υ may be dropped, and supplied or understood to be supplied at the end of the operations when needed to give homogeneity to an expression. Thus, for example,

$$(m + h\Upsilon)(m + k\Upsilon) = m^2 + (h + k)\Upsilon m + hk\Upsilon^2 = m^2 + (h + k)m + hk\Upsilon;$$

but this result may be obtained by the multiplication of $(m + h)(m + k)$, and supplying Υ (or imagining it to be supplied) to the final term in order to preserve the homogeneity of the form. In like manner, 0_ω or 0 with ω points over it may be used to denote the absolute zero of the order ω; but it will be more convenient to use the ordinary 0, having only recourse to the additional notation when thought necessary or desirable in order to make obvious the homogeneity of the terms in any equation or expression. Thus, for example, such an expression as $m^2 + 2bm + d = 0$, where m is a matrix, say of the 2nd

* Perhaps more advantageously by 1_ω. I shall hold myself at liberty in what follows to use whichever of these two notations may appear most convenient in any case as it arises.

order, and b and d monomials, set out in full would read $m^2 + 2bm + d\ddot{\Upsilon} = \ddot{0}$, meaning $m.m + 2\,bm + \begin{matrix} d & 0 \\ 0 & d \end{matrix} = \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix}$.

On the Inverse and Negative Powers of a Matrix.

The inverse of a matrix, denoted by m^{-1}, means the matrix which multiplied by m on either side produces multinomial unity. It is a matter of demonstration that when a matrix is non-vacuous (that is, has a finite content or determinant appertaining to it), an inverse to it fulfilling this *double* condition can always be found, and that if the product of mn is unity, so also must be that of nm.

It is a well-known fact, proved in the ordinary theory of determinants, that if every element in the first of two matrices is the logarithmic differential derivative, in respect to its correspondent in the second, of the content of that second, so conversely, every element of the second is the logarithmic derivative, in respect to its correspondent in the first, of the content of the first.

But two such matrices multiplied together in either sense would not give for their product multinomial unity; to obtain this product either matrix must be multiplied indifferently into or by the *transverse* of the other (meaning by the transverse of a matrix, the new matrix obtained by rotating the original one through 180° about its principal diagonal). In other words, if m be a given matrix and n be obtained from it by substituting for each element the logarithmic derivatives of its content in respect to its opposite, then $mn = \overset{\omega}{\Upsilon}$ and $nm = \overset{\omega}{\Upsilon}$, where ω means (as will always be the case throughout these lectures) the order of the matrices concerned. The n which satisfies these two equations (and it cannot satisfy the one without satisfying the other) will be called the inverse of m and be denoted by m^{-1}.

For brevity and suggestiveness it will be advantageous to write in general 1 for $\overset{\omega}{\Upsilon}$ as we write 0 for 0_ω, so that $mn = 1$ will imply $nm = 1 = mn$ and $n = m^{-1}$.

We may define in general (as in monomial algebra) m^{-i} to mean the inverse of m^i, that is, $(m^i)^{-1}$. We shall then have $(m^{-1})^i = m^{-i}$, for $mn.mn = 1$ implies $m.mn.n = mn = 1$ or $m^2n^2 = 1$. Hence $n^2 = m^{-2}$, that is, $(m^{-1})^2 = m^{-2}$. Also since $m^2n^2 = 1$, $m^3n^3 = mn = 1$ or $n^3 = m^{-3}$, that is, $(m^{-1})^3 = m^{-3}$, and so in general for all positive integer values of i, $(m^{-1})^i = m^{-i}$. And, as in monomial algebra, it may now be proved and taken as proved that, for all real values of i and j, whether positive or negative, $m^i.m^j = m^{i+j}$, and the same relation may be assumed to continue when i, j become general quantities. The elements in the inverse to any matrix m all involving the reciprocal of the determinant to m, if D be the content of m we may write $m^{-1} = \frac{1}{D}\mu$, where μ is a matrix all of whose elements are always finite. Hence we come to the important conclusion that for vacuous matrices inverses only exist in idea and are incapable of being realized so as to have an actual existence. In the sequel it will be shown that the inverse is only a single instance of an infinite class of matrices which exist ideally as functions of actual matrices, but are incapable of realization.

Suppose now that M, N are any two matrices such that $MN = 0$ or that $NM = 0$; multiplying each side of the equation by M^{-1} if such expression has an actual existence (that is, if M is non-vacuous), we obtain, from the known properties of zero, $N = 0$, but if M is vacuous no such conclusion can be drawn. So further if $m^i = 0$ (i being any positive integer), it will be seen under the third law of motion that m is necessarily vacuous. Hence from this equation it cannot be inferred that any lower power than the ith of m is necessarily zero.

On the Latent Roots and Different Degrees of Vacuity of Matrices.

If m be any matrix, the augmented matrix $m - \lambda\Upsilon$ or $m - \lambda \,.\, 1_\omega$ or $m - \lambda$ will be found simply by subtracting λ from each element in the principal diagonal of m. The content of this matrix or the same multiplied by -1 or any other constant, I term the latent function to m, which will be an algebraical function of the degree ω in λ (which may be termed the latent variable or carrier); and the ω roots of this function (that is, the ω values of the carrier which annihilate the latent function) I call the latent roots of the unaugmented matrix m. It is obvious from this definition that if λ_1 be any latent root of m, the content of $m - \lambda_1$ will vanish, that is, $m - \lambda_1$ will be vacuous, and conversely that if $m - \lambda_1$ is vacuous, λ_1 must be one of the latent roots to m. Thus if m is vacuous, one of the latent roots must be zero; if only one of them is zero I call m simply vacuous and say that its vacuity is 1: thus zero vacuity and simple vacuity mean the same thing as zero nullity and simple nullity respectively. More generally if any number i, but not $i+1$, of the latent roots of m are all of them zero, m will be said to have the vacuity i.

By a principal minor determinant to any matrix I mean any minor determinant whose matrix is divided by the principal diagonal into two triangles. It will then easily be seen that if s_i means in general the sum of the principal ith minors to m, and s_0 means the complete determinant, the assertion of m having the vacuity i is exactly coextensive with the assertion that

$$s_0 = 0, \quad s_1 = 0, \quad s_2 = 0, \ldots s_{i-1} = 0.$$

If the nullity of m is i, *every* qth minor of m is zero when $q < i$. Hence the vacuity cannot fall short of the nullity, but the converse is not true.

A matrix may not have any vacuity up to ω inclusive without the nullity being greater than 1. It will hereafter be shown, under the 2nd law of motion, that if $\lambda_1, \lambda_2, \ldots \lambda_\omega$ are the ω latent roots of m, then

$$(m-\lambda_1)(m-\lambda_2)\ldots(m-\lambda_\omega)=0 \text{ or say } M=0.$$

But it will be interesting even at this early stage to show that a theorem closely approaching this may be deduced from the distinction drawn between vacuous and non-vacuous matrices as regards their possession of real inverses.

I propose to prove instantaneously by this means that at all events $M^{\omega-1}=0$. It is obvious from any single instance of multiplication that mn and nm are not in general coincident. But if n could be expressed as a linear function of powers of m (including m^0 or 1_ω among such powers), mn and nm must be coincident. If now we take the ω^2 matrices

$$1, m, m^2, \ldots m^{\omega^2-1},$$

n at first blush one would say ought to be expressible as a linear function of these ω^2 quantities determinable by means of the solution of ω^2 linear equations, and can only escape being so expressible in consequence of the fact that these ω^2 powers of m are linearly related. Hence we must have an identical equation of the form

$$Am^{\omega^2-1}+Bm^{\omega^2-2}+Cm^{\omega^2-3}\ldots+Gm+H=0_\omega \text{ or say } Fm=0.$$

If now Fm were supposed to contain any factor other than

$$m-\lambda_1,\ m-\lambda_2, \ldots m-\lambda_\omega,$$

such factors being non-vacuous may be expelled from Fm; consequently the equation in question must be of the form

$$(m-\lambda_1)^{\alpha_1}(m-\lambda_2)^{\alpha_2}(m-\lambda_\omega)^{\alpha_\omega}=0,$$

and as the coefficients of the equation in m are necessarily rational we must have $\alpha_1=\alpha_2, \ldots, \alpha_\omega=\alpha$. Hence $\omega\alpha=\alpha_1+\alpha_2+\ldots\alpha_\omega<\omega^2$, and consequently $\alpha<\omega$.

Hence, at all events (since $M^{\omega-1-\theta}=0$ on multiplication by M^θ gives $M^{\omega-1}=0$),

$$\{(m-\lambda_1)(m-\lambda_2)\ldots(m-\lambda_\omega)\}^{\omega-1}=M^{\omega-1}=0. \quad \text{Q.E.D.}$$

LECTURE II.

On Reduction.

It follows from what has been already shown in Lecture I, when m is a matrix of the second order ($\omega-1$ being here unity) that $(m-\lambda_1)(m-\lambda_2)=0$. Understanding by m the matrix $\begin{matrix} t_1, & \tau_1, \\ t_2, & \tau_2, \end{matrix}$ the latent equation to m is

$$\begin{vmatrix} t_1-\lambda, & \tau_1 \\ t_2 \quad , & \tau_2-\lambda \end{vmatrix}=0,$$

that is, $$\lambda^2 - (t_1 + \tau_2)\lambda + (t_1\tau_2 - t_2\tau_1) = 0,$$

so that $$m^2 - (t_1 + \tau_2)m + (t_1\tau_2 - t_2\tau_1) = 0,$$

or, using the literation applied to the parametric triangle,

$$m^2 - 2bm + d = 0; \tag{1}$$

for since the content of $x + ym + zn$ is supposed to be

$$x^2 + 2bxy + 2cxz + dy^2 + 2eyz + fz^2,$$

that of $-\lambda + m$ will be found by making $z = 0$, $x = -\lambda$, $y = 1$. The variation of equation (1) obtained by taking ϵn for the increment of m (remembering that the variation of m^2 is $(m + \epsilon n)(m + \epsilon n) - m^2$, that is, $\epsilon(mn + nm)$) gives rise to the identical equation

$$mn + nm - 2bn - 2cm + 2e = 0, \tag{2}$$

and the variation of this again gives

$$n^2 + n^2 - 2cn - 2cn + 2f = 0,$$

or $n^2 - 2cn + f = 0$, as of course will be obtained immediately from (1) by substituting n, c, f in place of m, b, d.

The parameters c, f, if n represents $\begin{matrix} u_1 & v_1 \\ u_2 & v_2 \end{matrix}$ are the sum of the principal diagonal elements and the content of u, just as b, d are such sum and content in respect to m.

The parameter e (the connective to d and f) or rather its double $2e$ is obviously the emanant of d in respect to the operator

$$u_1\delta_{t_1} + u_2\delta_{t_2} + v_1\delta_{\tau_1} + v_2\delta_{\tau_2},$$

or, if we please, of f in respect to the inverse operator

$$t_1\delta_{u_1} + t_2\delta_{u_2} + \tau_1\delta_{v_1} + \tau_2\delta_{v_2},$$

that is, $$t_1v_2 + u_1\tau_2 - t_2v_1 - u_2\tau_1.$$

With the aid of the *catena* of equations in m, in m and n, and in n, any combination of functions of m and n may be reduced to the standard form

$$Amn + Bm + Cn + D.$$

For, in the first place,

$$\phi m = P(m^2 - 2bm + d) + rm + s = rm + s,$$

and similarly $$\psi n = \rho n + \sigma.$$

Hence the most general combination referred to is expressible as the product of alternating linear functions of m and n, and may therefore be reduced to a sum of terms of which each is a product of alternate powers of m and of n, each of which powers may again be reduced to the form of linear functions, and this process admits of being continually repeated.

Suppose then, at any stage of it, that the greatest number of occurrences of linear functions of m and n in the aggregate of terms is i; then at the next stage of the process the new aggregate will consist of monomial multipliers of one or more *simple* successions of m and n, and of terms in which the number of alternating linear functions never exceeds $i-1$; hence, eventually we must arrive at a stage when the aggregate will be reduced to a sum of monomial multipliers of simple successions of m and n, every such succession being of the form

$$(mn)^q \text{ or } m^{-1}(mn)^q \text{ or } (mn)^q n^{-1} \text{ or } m^{-1}(mn)^q n^{-1}.$$

But $\quad (mn)^2 = m\,.\,nm\,.\,n = -m(mn - 2bn - 2cm + 2e)\,n$

$$= -m^2n^2 + 2bmn^2 + 2cm^2n - 2emn$$
$$= -(2bm-d)(2cn-f) + 2bm(2cn-f) + 2c(2bm-d)\,n - 2emn$$
$$= -(2e-4bc)\,mn - df.$$

Hence $$(mn)^2 + 2(e-2bc)\,mn + df = 0.$$

Hence $\quad (mn)^q = P\{(mn)^2 + 2(e-2bc)\,mn + df\} + Amn + B = Amn + B,$

where A and B are known functions of $(e-2bc)$ and f; and therefore

$$m^{-1}(mn)^q = An + Bm^{-1} = An - \frac{B}{d}m + \frac{2Bb}{d}.$$

Similarly $$(mn)^q n^{-1} = Am - \frac{B}{f}n + \frac{2Bc}{f},$$

and $\quad m^{-1}(mn)^q n^{-1} = A + B(mn)^{-1} = -\frac{B}{df}mn + \left(A - B\frac{2e-4bc}{df}\right).$

And this being true (*mutatis mutandis*) for all values q, it follows that the function expressed by any succession of products of functions of m and n is reducible to the form of a linear expression in m, n, mn, in which the 4 monomial coefficients are known or determinable functions of the parameters to the corpus m, n.

The latent function to any such linear expression, say

$$Amn + Bm + Cn + D,$$

may be found in the same way as the latent function to mn has been found, namely, as follows:

$$(Amn + Bm + Cn + D)^2 = A^2(mn)^2 + AB(mnm + mmn) + AC(mnn + nmn)$$
$$+ 2ADmn + B^2m^2 + BC(mn + nm) + C^2n^2 + 2BDm + 2CDn + D^2$$
$$= A^2(-2e + 4bc)\,mn - A^2df + ABm(2bn + 2cm - 2e)$$
$$+ AC(2bn + 2cm - 2e)\,n + 2ADmn + B^2m^2 + BC(2bn + 2cm - 2e) + C^2n^2$$
$$+ 2BDm + 2CDn + D^2.$$

Let $\quad (Amn + Bm + Cn + D)^2 - 2P(Amn + Bm + Cn + D) + Q = 0$

be the identical equation to $Amn + Bm + Cn + D$.

The coefficient of mn in the development of the first term being

$$(4bc - 2e)\,A^2 + 2bAB + 2cAC + 2AD,$$

and m^2, n^2 being reducible to linear functions of m, n respectively, it follows that

$$P = A(2bc - e) + Bb + Cc + D.$$

To find Q it is only needful to fasten the attention upon the constant terms in the before named development reduced to the standard form. These will be

$$-A^2df - 2ABcd - 2ACbf - B^2d - 2BCe - C^2f + D^2, \text{ say } K,$$

and the constant part in $-2P(Amn+Bm+Cn+D)$ being $-2DP$, it follows that

$$\begin{aligned} Q &= 2AD(2bc-e) + 2BDb + 2CDc + D^2 - K \\ &= A^2df + 2ABcd + 2ACbf + 2AD(2bc-e) \\ &\quad + B^2d + 2BCe + C^2f + 2BDb + 2CDc, \end{aligned}$$

and consequently the latent function $\Lambda^2 - 2P\Lambda + Q$, of which the algebraical roots are the latent roots of $Amn + Bm + Cn + D$, is completely determined. Thus, for example, if the latent function of $m + n$ is required, making $A = D = 0$, $B = C = 1$, its value will be seen to be $\Lambda^2 - 2(b+c)\Lambda + d + 2e + f = 0$, so that the roots will be $b + c \pm \sqrt{\{(b+c)^2 - (d + 2e + f)\}}$.

On Involution.

In general, if m and n be two given binary matrices, and p any third matrix, say

$$m = \begin{matrix} t_1 & t_2 \\ t_3 & t_4 \end{matrix}, \quad n = \begin{matrix} \tau_1 & \tau_2 \\ \tau_3 & \tau_4 \end{matrix}, \quad p = \begin{matrix} T_1 & T_2 \\ T_3 & T_4 \end{matrix},$$

p may be expressed as a linear function of $\ddot{\Upsilon}$, m, n, mn or of $\ddot{\Upsilon}$, m, n, nm. For in order that p may be expressible under the form $A + Bm + Cn + Dnm$, observing that

$$nm = \begin{matrix} t_1\tau_1 + t_3\tau_2 & t_2\tau_1 + t_4\tau_2 \\ t_1\tau_3 + t_3\tau_4 & t_2\tau_3 + t_4\tau_4 \end{matrix},$$

and that $\ddot{\Upsilon} = \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}$, it is only necessary to write

$$\begin{aligned} A + Bt_1 + C\tau_1 + D(t_1\tau_1 + t_3\tau_2) &= T_1, \\ Bt_2 + C\tau_2 + D(t_2\tau_1 + t_4\tau_2) &= T_2, \\ Bt_3 + C\tau_3 + D(t_1\tau_3 + t_3\tau_4) &= T_3, \\ D + Bt_4 + C\tau_4 + D(t_2\tau_3 + t_4\tau_4) &= T_4, \end{aligned}$$

and then A, B, C, D may be found by the solution of these four linear equations: and this solution must always be capable of being effected unless the determinant

$$\begin{vmatrix} 1, & t_1, & \tau_1, & t_1\tau_1 + t_3\tau_2 \\ 0, & t_2, & \tau_2, & t_2\tau_1 + t_4\tau_2 \\ 0, & t_3, & \tau_3, & t_1\tau_3 + t_3\tau_4 \\ 1, & t_4, & \tau_4, & t_2\tau_3 + t_4\tau_4 \end{vmatrix}$$

vanishes.

When this is the case the matrices m, n, in the order in which they are written, will be said to be in sinistral involution. In like manner, if 1, n, m, mn are linearly related, m, n may be said to be in dextral involution. But it is very easy to see from the identical equation (2) that in this case these two involutions are really identical, for, since $A + Bm + Cn + Dmn = 0$, by subtraction

$$A + Bm + Cn - Dnm + 2Dcm + 2Dbn - 2De = 0,$$

that is, $$(A - 2eD) + (B + 2cD)\,m + (C + 2bD)\,n - Dnm = 0.$$

The above determinant then will be called the involutant to m, n or n, m, indifferently, for it will be seen, and indeed may be shown, *à priori*, that its value remains absolutely unaltered (not merely to a numerical factor *près*, but in sign and in arithmetical magnitude as well) when the Latin and Greek letters, or which is the same thing, when the matrices m and n are interchanged.

On the Linearform or Summatory Representation of Matrices, and the Multiplication Table to which it gives rise.

This method by which a matrix is robbed as it were of its areal dimensions and represented as a linear sum, first came under my notice incidentally in a communication made some time in the course of the last two years to the Mathematical Society of the Johns Hopkins University, by Mr C. S. Peirce, who, I presume, had been long familiar with its use. Each element of a matrix in this method is regarded as composed of an ordinary quantity and a symbol denoting its place, just as 1883 may be read

$$1\theta + 8h + 8t + 3u,$$

where θ, h, t, u, mean thousands, hundreds, tens, units, or rather, the places occupied by thousands, hundreds, tens, units, respectively.

Take as an example matrices of the second order, as

$$\begin{matrix} \alpha & \beta \\ \gamma & \delta \end{matrix} \qquad \begin{matrix} a & b \\ c & d. \end{matrix}$$

These may be denoted respectively by

$$\alpha\lambda + \beta\mu + \gamma\nu + \delta\pi, \quad a\lambda + b\mu + c\nu + d\pi;$$

their product by

$$(a\alpha + c\beta)\,\lambda + (b\alpha + d\beta)\,\mu + (a\gamma + c\delta)\,\nu + (b\gamma + d\delta)\,\pi,$$

which therefore must be capable of being made identical with

$$\begin{aligned} &a\alpha\lambda^2 + a\beta\lambda\mu + a\gamma\lambda\nu + a\delta\lambda\pi \\ &+ b\alpha\mu\lambda + b\beta\mu^2 + b\gamma\mu\nu + b\delta\mu\pi \\ &+ c\alpha\nu\lambda + c\beta\nu\mu + c\gamma\nu^2 + c\delta\nu\pi \\ &+ d\alpha\pi\lambda + d\beta\pi\mu + d\gamma\pi\nu + d\delta\pi^2, \end{aligned}$$

when a proper system of relations is established between the quadric combinations and the simple powers of λ.

The arguments of like coefficients in the two sums being equated together, there result the equations

$$\lambda^2 = \lambda, \quad \lambda\nu = \nu, \quad \mu\lambda = \mu, \quad \mu\nu = \pi,$$
$$\nu\mu = \lambda, \quad \nu\pi = \nu, \quad \pi\mu = \mu, \quad \pi^2 = \pi,$$

and again, the arguments to the 8 coefficients in the second sum which are not included among the coefficients of the first, being equated to zero, there result the equations

$$\lambda\mu = 0, \quad \lambda\pi = 0, \quad \mu^2 = 0, \quad \mu\pi = 0,$$
$$\nu\lambda = 0, \quad \nu^2 = 0, \quad \pi\lambda = 0, \quad \pi\nu = 0.$$

These 16 equalities may be brought under a single *coup d'œil* by the following multiplication table:

	λ	ν	μ	π
λ	λ	ν	0	0
ν	0	0	λ	ν
μ	μ	π	0	0
π	0	0	μ	π

In like manner it will be found that any matrix of the 3rd order as $\begin{matrix} a & b & c \\ d & e & f, \\ g & h & k \end{matrix}$

regarded as a quantity, may be expressed linearformly by the sum

$$a\lambda + b\mu + c\nu + d\pi + e\rho + f\sigma + g\tau + h\upsilon + k\phi,$$

where the topical symbols are subject to the multiplication table below written:

	λ	π	τ	μ	ρ	υ	ν	σ	ϕ
λ	λ	π	τ	0	0	0	0	0	0
π	0	0	0	λ	π	τ	0	0	0
τ	0	0	0	0	0	0	λ	π	τ
μ	μ	ρ	υ	0	0	0	0	0	0
ρ	0	0	0	μ	ρ	υ	0	0	0
υ	0	0	0	0	0	0	μ	ρ	υ
ν	ν	σ	ϕ	0	0	0	0	0	0
σ	0	0	0	ν	σ	ϕ	0	0	0
ϕ	0	0	0	0	0	0	ν	σ	ϕ

And, in like manner, matrices of any order ω may be expressed linearformly as the sum of ω^2 terms, each consisting of a monomial multiplier of a topical symbol, the entire ω^2 symbols being subject to a multiplication table containing ω^4 places, of which ω^3 will be occupied by the ω^2 simple symbols, each appearing ω times, and the remaining $\omega^4 - \omega^3$ places by the ordinary zero.

This conception applied to quadratic matrices might have served to establish the connection between them and Hamilton's quaternions, regarded as homogeneous functions of 1, i, j, k, themselves linear functions of the topical symbols λ, μ, ν, π; but the same result may be arrived at somewhat more simply by a method given in a *subsequent* lecture.

On the Corpus *formed by two Independent Matrices of the same order, and the Simple* Parameters *of such Corpus.*

By the latent function of a *corpus* (m, n) we may understand the content or any numerical multiplier of the content of (that is, the determinant to) the matrix $x + ym + zn$, where x, y, z are monomial carriers. This function will be a quantic of the order ω in x, y, z, and in the standard form the coefficient of x^ω may be supposed to be unity, so that it will contain $\frac{1}{2}(\omega^2+3\omega)$ coefficients, which may be termed the parameters of the corpus.

To fix the ideas, suppose $\omega = 3$ and let the latent function to

$$\begin{matrix} a & b & c & \alpha & \beta & \gamma \\ a' & b' & c' & \alpha' & \beta' & \gamma' \\ a'' & b'' & c'', & \alpha'' & \beta'' & \gamma'' \end{matrix}$$

be called F, where

$$F = x^3 + 3bx^2y + 3cx^2z + 3dxy^2 + 6exyz + 3fxz^2 + gy^3 + 3hy^2z + 3kyz^2 + lz^3.$$

Let m become $m + \epsilon n$, where ϵ is a monomial infinitesimal. Then the function to the corpus becomes the content of

$$x + y(m + \epsilon n) + zn, \text{ that is, } x + ym + (z + \epsilon y)n,$$

and consequently the variation of the function to (m, n) is $\epsilon y \delta_z F$. If then the *rate* of variation of any of the parameters, when n is the rate of variation of m, be denoted by prefixing to such parameter the symbol E, we shall find

$$Eb = c;\quad Ed = 2e;\quad Ee = f;\quad Eg = 3h;\quad Eh = 2k;\quad Ek = l;$$

and similarly, if $\mathfrak{F}$, preceding a parameter, be used to indicate its rate of variation corresponding to n's rate of variation being m, then

$$\mathfrak{F}c = b;\quad \mathfrak{F}f = 2e;\quad \mathfrak{F}e = d;\quad \mathfrak{F}l = 3k;\quad \mathfrak{F}k = 2h;\quad \mathfrak{F}h = g;$$

and the variations of c, f, l, as regards E, and of b, d, g, as regards $\mathfrak{F}$, are of course zero.

By forming the triangle of parameters

$$\begin{array}{c} 1 \\ b\ c \\ d\ e\ f \\ g\ h\ k\ l \\ p\ q\ r\ s\ t \end{array}$$

the law of variations of the parameters of the function to (m, n) (expressed in the ordinary manner by a ternary quantic affected with the proper numerical multipliers) becomes evident, whatever may be the order of the corpus (that is, of the matrices m and n, of which it is constituted): thus, for example, when $\omega = 4$, in addition to the previous expressions we shall find

$$Ep = 4q, \quad Eq = 3r, \quad Er = 2s, \quad Es = t, \quad Et = 0,$$
$$\mathfrak{E}t = 4s, \quad \mathfrak{E}s = 3r, \quad \mathfrak{E}r = 2q, \quad \mathfrak{E}q = p, \quad \mathfrak{E}p = 0.$$

By means of the above relations, any identical equation, into which enters one or more matrices, admits of being varied, so as to give rise to an identical equation connecting one additional number of the same.

Scholium.—In what precedes it will have been observed that the matter under consideration has always regard to matrices, or, as we may say, quantities of a fixed order ω, combined exclusively with one another and with ordinary monomial quantities. Every such combination forms as it were a *clausum* or world of its own, lying completely outside and having no relations with any other. It is, however, possible, and even probable, that as the theory is further evolved, this barrier may be found to give way and the worlds of all the various orders of quantity be brought into relation and intercommunion with one another.

LECTURE III.

On Quantity of the Second Order.

The theory of matrices of the second order seems to me to deserve a special preliminary investigation on various grounds. First, as affording a facile and natural introduction to the general theory (as the study of Conic Sections is usually made to precede that of universal Geometry); secondly, because it presents certain very special features distinguishing it from all other kinds of quantity, such as the coincidence of the two involutants (reminding one of the single image in the case of ordinary refraction as contrasted with the double image seen through iceland spar), or, again, the rational relation between the products of matrices of the second order, in whatever order the factors are introduced in the performance of the multiplication; and thirdly, because the theory of this kind of quantity has already been extensively studied and developed under the name or aspect of Quaternions. Hence it may not be out of place to make the remark that, as it surely would not be logical to seek for the origin of the conception included in the symbol $\sqrt{(-1)}$ in geometrical considerations, however important its application to geometrical exegesis, so now that an independent algebraical foundation has been discovered for the introduction and use of the symbols employed in Hamilton's theory, it would (it seems to me) be exceedingly illogical and contrary to good method to build the pure theory of the same upon space conceptions; the more so, as it will hereafter be shown that quantities of every order admit of being represented in a mode strictly analogous to that in which quantity of the second order is represented by quaternions, namely, if the order is ω, by ω^2-ions, or as I shall in future say, by *Ions*, of which the geometrical interpretation, although there is little doubt that it exists, is not yet discovered, and it must, it is certain, draw upon the resources of inconceivable space before it can be effected.