37.

NOTE ON SCHWARZIAN DERIVATIVES.

[*Messenger of Mathematics*, XV. (1886), pp. 74—76.]

READING with great pleasure and profit Mr Forsyth's masterly treatise on Differential Equations (in my opinion the best written mathematical book extant in the English language), it occurred to me to find an easy proof of the fundamental and striking identity concerning Schwarzian derivatives, from which all others are immediate consequences, namely $(y, x) - (z, x) = \left(\frac{dz}{dx}\right)^2 (y, z)$, where one of which is, it may be observed, that (y, x) like y'' has the property of remaining a factor of what it becomes when x and y are interchanged; a persistent factor, so to say, of its altered self. I will return to this point subsequently, my present concern is to give a natural proof of the above striking identity; to do this, it will be sufficient to show that (considering y, z, x, the two former as fixed, and the last as a variable function of a common variable) $\dfrac{(y, x) - (z, x)}{\left(\dfrac{dz}{dx}\right)^2}$ does not vary when x becomes $x + \epsilon\phi(x)$ where ϵ may be regarded as infinitesimal*. For then this must remain true by successive accumulation when x becomes any function whatever of itself, and accordingly making $x = z$ we obtain (y, z) as the value of the invariable quotient as was to be shown. Call† $\epsilon\delta\phi x = \theta$, then using dashes to denote differentiation *quâ* x, and a parenthesis to signify the augmented value of the derivatives, we obtain

$$(y') = y' - \theta y',$$
$$(y'') = y'' - 2\theta y'' - \theta' y',$$
$$(y''') = y''' - 3\theta y''' - 3\theta' y'' - \theta'' y'.$$

* It is easy to see *à priori* that if the theorem is true, it can only be so in virtue of (y, x) when x receives an infinitesimal, becoming of the form

$$(1 - 2\theta)(y, x) + \lambda\theta'',$$

as is subsequently shown to be the case in the text.

[† Cf. p. 306 below.]

Hence
$$(y'y''') = y'y''' - 4\theta y'y'' - 3\theta' y'y'' - \theta'' y'^2,$$
$$\tfrac{3}{2}(y''^2) = \tfrac{3}{2}y''^2 - 6\theta y''^2 - 3\theta' y'y'',$$
$$(y')^2 = y'^2 - 2\theta y'^2,$$
$$((y, x)) = (1 + 2\theta)\{(y, x) - 4\theta (y, x)\} - \theta''$$
$$= (1 - 2\theta)(y, x) - \theta'',$$

and
$$((y, x) - (z, x)) = (1 - 2\theta)[(y, x) - (z, x)].$$

Hence
$$\left(\frac{(y, x) - (z, x)}{\left(\dfrac{dz}{dx}\right)^2}\right) = \frac{(y, x) - (z, x)}{\left(\dfrac{dz}{dx}\right)^2},$$

that is, the right-hand expression does not change, when y, z remaining fixed forms of function, x passes from one form of function of the independent variable to another; as was to be shown.

From what precedes, it appears that if y, z, x be regarded as functions of t, then $\{(y, x) - (z, x)\}\left(\dfrac{dx}{dt}\right)^2$ is a constant function in the sense that it remains unaltered, whatever function x may be of t, or which is the same thing if y and z functions of x when expressed as functions of x' (any function of x) are written y', z', then $(y', x') - (z', x')$ is identical with $(y, x) - (z, x)$, save as to a *factor* which depends only on the form of the *substitution* of x' for x. Hence to all intents and purposes, any function of the differences of the Schwarzian derivatives of any system of functions of the same variable, in respect thereto, is (in a sense comprising, but infinitely transcending the sense in which that word is used in Algebra) a *covariant* of the system.

ADDENDUM.—Let us for the moment call functions of x, y which either remain unaltered or only change their sign when x and y are interchanged self-reciprocating functions.

The first case of the kind is $\dfrac{y''}{y'^{\frac{3}{2}}}$, the next is $\dfrac{y'y''' - y''^2}{y'^2}$, and obviously a very general one of this sort will be the function

$$\left(\frac{1}{y'^{\frac{1}{2}}}\frac{d}{dx}\right)\log y'.$$

For greater simplicity, let us call the *numerator* of any such function when expanded and brought to the lowest possible common denominator, a *reciprocant*, the highest index of differentiation which such reciprocant contains its *order*, and the number of factors in each term its *degree*. Then in any reciprocant so formed the degree is always just one unit less than the order: but as a matter of fact the function so obtained is in general not irreducible, so that its degree may be depressed, and it becomes a question of much interest to form the scale of degrees of reciprocants of this sort. For the orders 2, 3, 4, 5, 6 the degrees in question are respectively 1, 2, 2, 3, 3. Calling the successive derivatives of y, a, b, c, d, ..., they will be found to be

$$\begin{aligned}
&a,\\
&b,\\
&2ac - 3b^2,\\
&ad - 5bc,\\
&2a^2e - 15acd - 10ad^2 + 35b^2c,\\
&2a^2f - 21abc - 35acd + 60ab^2d + 110bc^2,
\end{aligned}$$

where each form is obtained by operating upon the preceding one with the operator $a(b\delta c + c\delta d + d\delta e + \ldots) - \lambda b$ (λ meaning half the weight + the degree of the operand), combining the result of this operation in each *alternate* case with a *legitimate* combination of those that precede, and in that case dividing out by a. I have proved that in this way can be obtained an infinite progression of reciprocants, of which the leading terms (substituting numbers for letters), will be alternately of the forms $1^i.(2i+1)$ and $1^i.(2i+2)$. Every other reciprocant can be formed algebraically from these primordial forms, as every seminvariant can be obtained from the primordial forms a, $ac - b^2$, $a^2d - 3abc + 2b^3$, ... The two theories run in parallel courses, but their relationship is that which naturalists call homoplazy as distinguished from homogeny; I propose to give further developments of this new algebraical theory in a subsequent Note.